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NEW CLASS OF INVARIANT AND PARTIALLY INVARIANT SOLUTIONS TO THE ONE-DIMENSIONAL GAS DYNAMICS EQUATIONS

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The presentation is devoted to discussing a new class of solutions of the one-dimensional gas dynamics equations

$$\begin{array}{l} u_t + uu_x + \tau p_x = 0, \\ \tau_t + u\tau_x - \tau u_x = 0, \\ p_t + up_x + A(\tau, p)u_x = 0. \end{array} \tag{1}$$

Here ρ is the density, u is the velocity, p is the pressure, and c is the sound speed ($c^2 = \tau A$). For a polytropic gas $A = \gamma p$, $\gamma > 1$. The new class of solutions is obtained by applying sequentially the method differential of constraints and group analysis.

The idea of the method of differential constraints was proposed by N. N. Yanenko [1] as a generalization of solutions with degenerated hodograph. This class of solutions is characterized by finite relations between dependent functions. Well-known classes of such solutions are simple and double waves. A survey of the method can be found in [2,3]. Applications of the method of differential constraints to the one-dimensional gas dynamics equations written in Lagrangian coordinates are given in [4, 5], and in the Eulerian coordinates in [6].

Another approach for generalizing the set of solutions with a degenerated hodograph is given by L. V. Ovsiannikov [7, 8]. He extended a set of invariant solutions by introducing the notion of a partially invariant solution. Invariant and partially invariant solutions of the gas dynamics equations were studied in program PODMODELI [9]. Review of the results of this program can be found in [10].

The problem of relations between partially invariant solutions and solutions obtained by the method of differential constraints was repeatedly set up by L. V. Ovsiannikov and N. N. Yanenko. In the presentation it is given answer to this problem for a particular class of solutions of the gas dynamics equations.

Let us consider solutions of (1) which are defined by the differential constraints

$$\tau_x = \varphi^\tau(\tau, p), \quad p_x = \varphi^p(\tau, p), \quad u_x = \varphi^u(\tau, p). \tag{2}$$

The functions $\varphi^\tau(\tau, p)$, $\varphi^p(\tau, p)$, $\varphi^u(\tau, p)$ have to satisfy the equations

$$\begin{array}{c} \varphi^u(\frac{\gamma p}{\tau}\varphi^\tau_p - \varphi^\tau_\tau) + \varphi^u_p\varphi^p + \varphi^u_\tau\varphi^\tau = 0, \\ \tau\varphi^u(\varphi^u_p\frac{\gamma p}{\tau} - \varphi^u_\tau) - \tau(\varphi^p\varphi^p_p + \varphi^\tau\varphi^p_\tau) = \varphi^{u\,2} + \varphi^p\varphi^\tau, \\ \tau\varphi^u(\varphi^p_p\frac{\gamma p}{\tau} - \varphi^p_\tau) - \gamma p(\varphi^p\varphi^u_p + \varphi^\tau\varphi^u_\tau) = (\gamma+1)\varphi^p\varphi^u. \end{array} \tag{3}$$

Notice that if $\Delta = \tau_x p_t - \tau_t p_x = -\tau\varphi^u(\varphi^p + \frac{\gamma p}{\tau}\varphi^\tau) \neq 0$, then from the relations $\tau = \tau(t, x)$ and $p = p(t, x)$ one can find $t = t(\tau, p)$, $x = x(\tau, p)$. Substituting them into the values for the derivatives $\tau_x(t, x)$, $p_x(t, x)$, $u_x(t, x)$, one finds that all solutions of the gas dynamics equations with $\Delta \neq 0$ can be described by the differential constraints (2). If the functions $\varphi^\tau(\tau, p)$, $\varphi^p(\tau, p)$, $\varphi^u(\tau, p)$ are found, then a solution of the gas dynamics equations (1) is restituted by quadratures. Thus, for finding exact solutions of the gas dynamics equations one can use solutions of system (3).

Equations (2) admit the Lie algebra with the generators

$$X_1 = \tau\partial_\tau + p\partial_p, \quad X_2 = \varphi^\tau\partial_{\varphi^\tau} + \varphi^p\partial_{\varphi^p} + \varphi^u\partial_{\varphi^u},$$
$$X_3 = \tau\partial_\tau - p\partial_p + \varphi^\tau\partial_{\varphi^\tau} - \varphi^p\partial_{\varphi^p}.$$

The generators X_2 and X_3 are inherited by the operators admitted by the one-dimensional gas dynamics equations of a polytropic gas (1): $Y_1 = t\partial_t + x\partial_x$, $Y_2 = \tau\partial_\tau - p\partial_p$, respectively. The generator X_1 produces a new set of symmetries: it is not admitted by a system of the gas dynamics equations (1).

The algebra $\{X_1, X_2, X_3\}$ is Abelian. In the presentation all classes of invariant and partially invariant solutions related with the subalgebra $X_1 + k_2X_2 + k_3X_3$ are considered. Here k_2 and k_3 are constant.

It is interesting to note that among the set of invariant solutions there is one class of solutions which has a functional arbitrariness. It is interesting because usually for two independent variables an invariant solution with respect to a one-parameter Lie group is reduced to a system of ordinary differential equations which has only constant arbitrariness.

For partially invariant solutions it is shown that all unreduceable partially invariant solutions coincide with solutions characterized by two differential constraints of first-order of the gas dynamics equations [6]. Thus, this gives a solution of the problem which was set up by N. N. Yanenko and L. V. Ovsiannikov.

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