Международная конференция "Дифференциальные уравнения, теория функций и приложения", 2007, с. 408–409

УДК 517.95

THE DIRICHLET PROBLEM FOR ANISOTROPIC QUASILINEAR DEGENERATE ELLIPTIC EQUATIONS

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Consider the Dirichlet boundary value problem

$$-\sum_{i=1}^{n} \mu_i(|u_{x_i}|^{p_i} u_{x_i})_{x_i} = c(\mathbf{x})g(u) + f(\mathbf{x}) \ \text{ in } \ \Omega \subset \mathbf{R^n}, \quad u = 0 \ \text{ on } \ \partial\Omega. \tag{1}$$

Here constants $\mu_i > 0$ and $p_i \geq 0$. We assume that the parts of $\partial\Omega$ lying in the half spaces $x_i \leq 0$ and $x_i \geq 0$ can be expressed as

$$x_i = F_i \ \text{ and } \ x_i = G_i, \quad i = 1, \dots, n,$$

respectively, where F_i and G_i are independent of x_i. Without loss of generality we suppose that

$$\Omega \subset \{\mathbf{x} : -l_i \leq x_i \leq l_i, \quad i = 1, \dots, n\}.$$

Concerning the function g we suppose that

$$g(0) = 0, \quad g(z) > 0 \ \text{ if } \ z > 0 \ \text{ and } \ |g(z)| \leq g(C) \ \text{ for } \ |z| \leq C, \tag{2}$$

where C is an arbitrary positive constant.

Definition 1. We say that $u(\mathbf{x})$ is *a generalized solution of problem* (1) if $u(\mathbf{x}) \in W^{1,\infty}(\Omega)$, $u(\mathbf{x}) = 0$ for $\mathbf{x} \in \partial\Omega$ and

$$\int_\Omega \sum_{i=1}^{n} \mu_i |u_{x_i}|^{p_i} u_{x_i} \phi_{x_i}(\mathbf{x}) d\mathbf{x} = \int_\Omega \Big(c(\mathbf{x})g(u) + f(\mathbf{x}\Big)\phi(\mathbf{x}) d\mathbf{x} \ \ \forall \phi \in \overset{0}{W}{}^{1,r}(\Omega), \quad 1 \leq r < +\infty.$$

New a priori estimates for solutions and for the gradient of solutions of problem (1) are established. Based on these estimates sufficient conditions guaranteeing the solvability of the problem are formulated. The results are new even in the semilinear case when the principal part is the Laplace operator.

Assume that there exists a positive constant M such that

$$(c_0 g(M) + f_0)\Big(\frac{3l^2 + 2l}{2}\Big)^{p+1} < \mu(p+1)M^{p+1}. \tag{3}$$

Here $p = p_{i_0} = \max\{p_1, \dots, p_n\}$, $\mu = \mu_{i_0}$, $l = l_{i_0}$, $c_0 = \sup_\Omega |c(\mathbf{x})|$ and $f_0 = \sup_\Omega |f(\mathbf{x})|$.

Theorem 1. i) *Suppose that* $c(\mathbf{x})$ *and* $f(\mathbf{x})$ *are bounded in* $\bar{\Omega}$, $g(u)$ *is a Hölder continuous function on* $[-M, M]$ *and conditions* (2), (3) *are fulfilled. If* $\Omega \subset \mathbf{R}^n$ *is strictly convex, then there exists a generalized solution of problem* (1) *such that*

$$||u||_{L_\infty(\Omega)} \leq M_0 \ \ \textit{and} \ \ ||u_{x_i}||_{L_\infty(\Omega)} \leq (1 + 2l_i)\Big(\frac{\Phi_0}{\mu_i(1 + p_i)}\Big)^{\frac{1}{p_i+1}}, \quad i = 1, \dots, n,$$

where $M_0 = \inf\{M : M \ \ satisfies \ \ (3)\}$ *and*

$$\Phi_0 = \max_{\bar{\Omega}\times[-M_0,M_0]} |c(\mathbf{x})g(u) + f(\mathbf{x})|.$$

ii) *If in addition* $c(\mathbf{x}) \le 0$ *and* $g(u)$ *is a nondecreasing function then the solution is unique.*

Remark 1. If $g(u) = u^q$ (or $g(u) = |u|^{q-1}u$ or $g(u) = |u|^q$) and $p+1 > q$ as well as if $c_0 = 0$ then for an arbitrary bounded $f(\mathbf{x})$ one can always find $M > 0$ satisfying (3) and as a consequence obtain the existence of a generalized solution by Theorem 1.

Consider the Dirichlet problem for semilinear equation:

$$-\mu\Delta u = c(\mathbf{x})g(u) + f(\mathbf{x}) \ \text{ in } \ \Omega \subset \mathbf{R^n} \quad u = 0 \ \text{ on } \ \partial\Omega. \tag{4}$$

Assume that there exists a positive constant M such that

$$(c_0 g(M) + f_0)\frac{3\tilde{l}^2 + 2\tilde{l}}{2} < \mu M, \quad \tilde{l} = \min\{l_1, \dots, l_n\}. \tag{5}$$

Theorem 2. i) *Suppose that* $c(\mathbf{x}), f(\mathbf{x}) \in C^\gamma(\bar{\Omega})$, $g(u) \in C^\gamma([-M, M])$, $\partial\Omega \in C^{2+\gamma}$, $\gamma \in (0,1)$. *If* (5) *is fulfilled then there exists a classical solution of problem* (4) $u(x) \in C^{2+\gamma}(\bar{\Omega})$ *such that*

$$\max_{\Omega} |u| \le M_0, \quad \textit{where} \quad M_0 = \inf\{M : M \quad \textit{satisfies} \quad (5)\}$$

ii) *If in addition* $c(\mathbf{x}) \le 0$ *and* $g(u)$ *is a nondecreasing function then the solution is unique.*

Remark 2. If Ω in Theorem 2 is strictly convex then we additionally have

$$\max_{\Omega} |u_{x_i}| \le (1 + 2l_i)\frac{\Phi_0}{\mu}, \quad i = 1, \dots, n.$$