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NEW PROPERTY OF PDE AND CONSTRUCTION OF SOLUTION TO PDE IN THE PARAMETRIC FORM

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The new property of PDE is found. It allows to construct new exact solutions in parametric form. This property is spread to the PDE with variable coefficients. The algorithm is stated in the conditional form. Let's assume that all necessary functions exist and have that smoothness which is required for the algorithm can be applied. Consider the quasilinear hyperbolic equation

$$\mu Z_{tt} + Z_t - (K(Z)Z_x)_x + F(Z) = 0, \tag{1}$$

where μ is the parameter. Let's make a change of variables $Z(x,t)\big|_{x=x(\xi,\delta),t=t(\xi,\delta)} = U(\xi,\delta)$ where function $Z(x,t)$ is solution of the equation (1). Let's assume that Jacobian $\det J = x_\xi t_\delta - x_\delta t_\xi \neq 0$ is not equal to zero. Let's assume that there exists an inverse mapping, at least locally, $\xi = \xi(x,t)$, $\delta = \delta(x,t)$, the derivatives of direct and inverse mapping have the form $\frac{\partial x}{\partial \xi} = \det J \frac{\partial \delta}{\partial t}$, $\frac{\partial t}{\partial \xi} = -\det J \frac{\partial \delta}{\partial x}$, $\frac{\partial x}{\partial \delta} = -\det J \frac{\partial \xi}{\partial t}$, $\frac{\partial t}{\partial \delta} = \det J \frac{\partial \xi}{\partial x}$. Let's assume that the relations for flows have the form $K(Z)\frac{\partial Z}{\partial x}\big|_{x=x(\xi,\delta),t=t(\xi,\delta)} = Y(\xi,\delta)$, $K(Z)\frac{\partial Z}{\partial t}\big|_{x=x(\xi,\delta),t=t(\xi,\delta)} = T(\xi,\delta)$. We obtain

$$K(U)\Big(\frac{\partial U}{\partial \xi}\frac{\partial t}{\partial \delta} - \frac{\partial U}{\partial \delta}\frac{\partial t}{\partial \xi}\Big) = Y(\xi,\delta)\det J, \qquad K(U)\Big(-\frac{\partial U}{\partial \xi}\frac{\partial x}{\partial \delta} + \frac{\partial U}{\partial \delta}\frac{\partial x}{\partial \xi}\Big) = T(\xi,\delta)\det J. \tag{2}$$

The equation (1) after all substitutions has the form

$$\mu K(U)\Big(\frac{\partial (T/K)}{\partial \delta}\frac{\partial x}{\partial \xi} - \frac{\partial (T/K)}{\partial \xi}\frac{\partial x}{\partial \delta}\Big) - K(U)\Big(\frac{\partial Y}{\partial \xi}\frac{\partial t}{\partial \delta} - \frac{\partial Y}{\partial \delta}\frac{\partial t}{\partial \xi}\Big) + \det J[T + K(U)F(U)] = 0. \tag{3}$$

The equality for mixed derivative has to be true $Z_{x,t} = Z_{t,x}$. This relation can be rewritten as follows

$$-\frac{\partial}{\partial \xi}\Big[\frac{Y}{K(U)}\Big]\frac{\partial x}{\partial \delta} + \frac{\partial}{\partial \delta}\Big[\frac{Y}{K(U)}\Big]\frac{\partial x}{\partial \xi} - \frac{\partial}{\partial \xi}\Big[\frac{T}{K(U)}\Big]\frac{\partial t}{\partial \delta} + \frac{\partial}{\partial \delta}\Big[\frac{T}{K(U)}\Big]\frac{\partial t}{\partial \xi} = 0. \tag{4}$$

The analysis of the system (2)–(4) is divided into two stages. At the first stage we shall consider (2)–(4) as the nonlinear algebraic system for the derivatives x'_ξ, x'_δ, t'_ξ, t'_δ.

Theorem 1. *The nonlinear algebraic system* (2) – (4) *for the derivatives* x'_ξ, x'_δ, t'_ξ, t'_δ *has unique solution of the form*

$$\begin{aligned}\frac{\partial x}{\partial \xi} =& [K[-FK^3U'_\xi[U'_\delta T'_\xi - T'_\delta U'_\xi] - \\ & K^2[-TU'_\delta Y'^2_\xi - TT'_\delta U'^2_\xi + TU'_\delta T'_\xi U'_\xi - YY'_\delta T'_\xi U'_\xi + TY'_\delta Y'_\xi U'_\xi - TU'_\delta Y'^2_\xi] + \\ & \mu[T^2K'U'_\xi[U'_\delta T'_\xi - U'_\xi T'_\delta] - KTT'_\xi[T'_\xi U'_\delta - U'_\xi T'_\delta]]]]/P_1(\xi,\delta) = g_1(\xi,\delta),\end{aligned}$$

$$\frac{\partial x}{\partial \delta} = -\,[K[FU'_\delta[U'_\delta T'_\xi - T'_\delta U'_\xi]K^3 + K^2[TT'_\xi U'^2_\delta - YY'_\delta T'_\xi U'_\delta - TT'_\delta U'_\xi U'_\delta + YT'_\delta Y'_\xi U'_\delta - TY'_\delta Y'_\xi U'_\delta + TY'^2_\delta U'_\xi] + \mu[-K'U'_\delta[U'_\delta T'_\xi - U'_\xi T'_\delta]T^2 - KT'_\delta[T'_\delta U'_\xi - U'_\delta T'_\xi]T]]]/P_1(\xi,\delta) = g_2(\xi,\delta),$$

$$\frac{\partial t}{\partial \xi} = [K[FU'_\xi[U'_\delta Y'_\xi - Y'_\delta U'_\xi]K^3 - K^2[YY'_\xi - TU'_\xi][U'_\delta Y'_\xi - Y'_\delta U'_\xi] + \mu[K[YU'_\delta T'^2_\xi - YU'_\xi T'_\delta T'_\xi - TU'_\xi T'_\xi Y'_\delta + TU'_\xi T'_\delta Y'_\xi] - T^2K'U'_\xi[U'_\delta Y'_\xi - U'_\xi Y'_\delta]]]]/P_1(\xi,\delta) = g_3(\xi,\delta),$$

$$\frac{\partial t}{\partial \delta} = [K[FU'_\delta[U'_\delta Y'_\xi - Y'_\delta U'_\xi]K^3 + K^2[YY'_\delta - TU'_\delta][Y'_\delta U'_\xi - U'_\delta Y'_\xi]K^2 + \mu[K[YT'_\delta[U'_\delta T'_\xi - U'_\xi T'_\delta] + TU'_\delta[Y'_\xi T'_\delta - T'_\xi Y'_\delta]] - T^2K'U'_\delta[Y'_\xi U'_\delta - U'_\xi Y'_\delta]]]]/P_1(\xi,\delta) = g_4(\xi,\delta),$$

where $P_1(\xi,\delta)) = [F[-YU'_\delta T'_\xi + YT'_\delta U'_\xi - TY'_\delta U'_\xi + TU'_\delta Y'_\xi]K^3 - [Y'_\delta U'_\xi T^2 - U'_\delta Y'_\xi T^2 + YTU'_\delta T'_\xi - YTT'_\delta U'_\xi - Y^2Y'_\delta T'_\xi + Y^2T'_\delta Y'_\xi]K^2 + \mu[T^2[YK'[U'_\delta T'_\xi - U'_\xi T'_\delta] + K[T'_\delta Y'_\xi - Y'_\delta T'_{xi}]] - T^3K'[U'_\delta Y'_\xi - Y'_\delta U'_\xi]]$.

At the second stage we shall formulate a condition of solvability of the system for the functions $x = x(\xi,\delta)$, $t = t(\xi,\delta)$. The key point is following statement:

Theorem 2. *The necessary conditions of solvability for system are the equalities*

$$(x''_{\xi\delta} - x''_{\delta\xi})/T = 0, \qquad (t''_{\xi\delta} - t''_{\delta\xi})/Y = 0.$$

These two equalities **coincide** for any smooth functions $Y(\xi,\delta)$, $T(\xi,\delta)$, $K(U)$, $F(U)$, this gives the relation for the functions $U(\xi,\delta)$, $Y(\xi,\delta)$, $T(\xi,\delta)$

$$\frac{\partial}{\partial \delta} g_1(\xi,\delta) - \frac{\partial}{\partial \xi} g_2(\xi,\delta) = 0. \tag{5}$$

This new property of PDE allows **to construct new solutions in the parametric form of the equation (1)** [1–3].

Let's discuss one possible way how to satisfy a common relation (5).

Let's find functions Y, T in (5) in the form $Y(\xi,\delta) = r(U) + h(U)G(\xi,\delta,U)$, $T(\xi,\delta) = w(U) + v(U)G(\xi,\delta,U)$.

This equality is possible to satisfy having equated to with zero coefficients at degrees G. We have the system of four equations on four functions r, h, w, v. The functions $G(\xi,\delta,U)$, U (!) remain arbitrary.

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