Международная конференция "Дифференциальные уравнения, теория функций и приложения", 2007, с. 534–535

УДК 517.9; 539.3

PARTIAL DIFFERENTIAL EQUATIONS ON HYPERSURFACES AND THE SHELL THEORY

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Consider a hypersurface $\mathcal{S}$ be given by a local diffeomorphism $\Theta : \Omega \to \mathcal{S}$, $\Omega \subset \mathbb{R}^{n-1}$, which has the maximal rang $n-1$ and $\nu(\mathcal{X}) = (\nu_1(\mathcal{X}), \dots, \nu_j(\mathcal{X}))^\top$ be the outer unit normal vector to $\mathcal{S}$ at $\mathcal{X} \in \mathcal{S}$. We propose to write partial differential equations on hypersurface in cartesian coordinates of the ambient space instead of more customary local coordinates and Riemannian metric tensor of the underlying surface. This seemingly trivial idea simplifies the form of many classical differential equations on the surface. namely, the surface gradient, which maps a function $f \in \mathbb{C}^1(\mathcal{S})$ to the tangent vector fields to $\mathcal{S}$ is written in the form

$$\nabla_{\mathcal{S}} = \mathcal{D}_{\mathcal{S}} := (\mathcal{D}_1, \dots, \mathcal{D}_n)^\top , \tag{1}$$

while the formally adjoint operator-the surface divergence which maps a smooth tangent vector field $U = \sum_{j=1}^n U^j \partial/\partial x_j \in \mathcal{TS}$ to scalar function-in the form:

$$\mathrm{Div}_{\mathcal{S}} U = \sum_{j=1}^n \mathcal{D}_j \mathrm{U}^j , \tag{2}$$

Here $\mathcal{D}_j := \partial_j - \nu_j(\mathcal{X})\nabla_\nu$ denotes the covariant Günter's derivative, which is tangential to the surface and $\nabla_\nu := \sum_{j=1}^n \nu_j \partial_j$ is the normal derivative.

Respectively, the Laplace – Beltrami operator acquires the form

$$\boldsymbol{\Delta}_{\mathcal{S}}\, \varphi := \mathrm{Div}_{\mathcal{S}} \nabla_{\mathcal{S}} \varphi = \sum_{j=1}^n \mathcal{D}_j^2 \varphi = \sum_{1 \le j < k \le n} \mathcal{M}_{jk}^2 \varphi , \qquad \varphi \in \mathbb{C}^2(\mathcal{S}) , \tag{3}$$

where $\mathcal{M}_{jk} := \nu_j \partial_k - \nu_k \partial_j$ are the Stoke's derivatives (natural entries of the Stoke's formulae).

Based on the principle that, at equilibrium, the displacement minimizes the total free (elastic) energy

$$\mathcal{E}[U] := -\frac{1}{2} \int_{\mathcal{S}} E(x, \nabla U(x))\, dS, \qquad U \in \mathcal{TS} , \tag{4}$$

we derive the following expression for the Lamé operator $\mathcal{L}$ on $\mathcal{S}$

$$\mathcal{L}_{\mathcal{S}} = -2\mu\, \mathrm{Def}_{\mathcal{S}}^* \mathrm{Def}_{\mathcal{S}} + \lambda\, \nabla\, \mathrm{Div}_{\mathcal{S}} , \tag{5}$$

where $(\mathrm{Def}_{\mathcal{S}}\, U)(V, W) = \frac{1}{2}\Big\{ \langle \nabla_V^{\mathcal{S}} U, W \rangle + \langle \nabla_W^{\mathcal{S}} U, V \rangle \Big\}$ denotes the deformation tensor and $\nabla_W^{\mathcal{S}} U$ — the covariant derivative of tangent vector fields $V, W \in \mathcal{TS}$. Representation in terms of Günter's derivatives has the form

$$\mathcal{L}_{\mathcal{S}} U = \mu\, \pi_{\mathcal{S}}\, \mathrm{Div}_{\mathcal{S}}\, \nabla_{\mathcal{S}} U + (\lambda + \mu)\, \nabla_{\mathcal{S}} \mathrm{Div}_{\mathcal{S}} U - \mu\, (n-1) \mathcal{H}_{\mathcal{S}} \mathcal{W}_{\mathcal{S}} \mathrm{U} . \tag{6}$$

$\lambda, \mu \in \mathbb{R}$ are the Lamé moduli, whereas $\mathcal{H}_{\mathcal{S}} := (n-1)^{-1} \sum_{j=1}^n \mathcal{D}_j \nu_j$, $\mathcal{W}_{\mathcal{S}} = \big[\mathcal{D}_j \nu_k\big]_{n \times n}$ stand, respectivthe surface. $\pi_{\mathcal{S}} : \mathbb{R}^n \to \mathcal{TS}$ is the orthogonal projection onto the tangent hyperspace to $\mathcal{S}$.

A special accent is made on a thin flexural shell problems in elasticity. We suggest for their study the above Lamé equation which describe displacement of the mid-surface.

We dwell on the approach that considers flexural shells as two-dimensional deformable bodies in the spirit of Cosserat shell theories.

There exist a number of approaches proposed for modeling linearly elastic flexural shells. Started by the Cosserats pioneering work (1909), Goldenveiser (1961), Naghdi (1963), Vekua (1965), Novozhilov (1970), Koiter (1970) and many others contributed essentially the development of the shell theory. Ellipticity of the corresponding partial differential equations was proved much later by the different authors: Roug'e (1969), Coutris (1973), Gordeziani (1974), Shoikhet (1974), Ciarlet & Miara (cf. [1] for survey and further references).

Relatively simple form of operators recorded in terms of Günter's and Stoke's derivatives (1)–(6) enable simplified treatment of corresponding boundary value problems. For this purpose we apply the lax-Milgram lemma which require proofs of Korn's inequalities first without and later with boundary conditions. Solutions of the homogeneous equation $\mathrm{Def}_{\mathcal{S}} U = 0$, which are also solve the homogeneous Lamé equation $\mathcal{L}_{\mathcal{S}} U = 0$ and are known as the Killing's vector fields, are investigated. The investigation is based on the representation of the deformation tensor $\big[\mathfrak{D}_{jk}(U)\big]_{n\times n}$ in the terms of Günter's derivatives has the form

$$\mathfrak{D}_{jk}(U) = \frac{1}{2}\Big[\mathcal{D}_k U_j + \mathcal{D}_j U_k + \nabla_U\big(\nu_j \nu_k\big)\Big], \qquad U = \sum_{j=1}^{n} U^j \frac{\partial}{\partial x_j} \in \mathcal{TS}.$$

The above results are partly published in the paper [2].

REFERENCES

1. *Ciarlet P. G.* Mathematical Elasticity. V. III: Theory of Shells // Studies in Mathematics and Applications. V. 29. Elsevier, North-Holland, Amsterdam, 2000.
2. *Duduchava R., Mitrea D., Mitrea M.* Differential operators and boundary value problems on surfaces // Mathematische Nachrichten. 2006. V. 9–10. P. 996–1023.