Международная конференция "Дифференциальные уравнения, теория функций и приложения", 2007, с. 538–539

УДК 539.3

A BOUNDARY VALUE PROBLEM FOR AN INFINITE ELASTIC STRIP WITH A SEMI-INFINITE CRACK

© Hiromichi Itou * and Atusi Tani **

* h-itou@math.sci.gunma-u.ac.jp, ** tani@math.keio.ac.jp

* *Gunma University, Kiryu, Japan;*
** *Keio University, Yokohama, Japan*

In this talk we study a boundary value problem in a two-dimensional infinite elastic strip with a semi-infinite crack (cf. [7] in References).

By $u = (u_i)_{i=1,2,3}$ and $\sigma = (\sigma_{ij})_{i,j=1,2,3}$ we denote the displacement vector and the stress tensor, respectively. Let

$$\Omega = \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in \mathbb{R}, -a < x_2 < a\} \ (a > 0)$$

be a strip in $\mathbb{R}^2$, representing a homogeneous isotropic elastic plate in the state of a plane strain. Then, the linearized elasticity equations for a homogeneous isotropic material consist of the constitutive law (Hooke's law) and the equilibrium conditions without any body forces. On the boundaries of the strip

$$\partial\Omega_+ = \{(x_1, a) \mid x_1 \in \mathbb{R}\} \quad \text{and} \quad \partial\Omega_- = \{(x_1, -a) \mid x_1 \in \mathbb{R}\}$$

Neumann and Dirichlet boundary conditions are imposed, respectively. We denote by

$$\Gamma = \{(x_1, 0) \mid -\infty < x_1 \le 0\}$$

the crack in Ω. On the crack we assume the free traction condition.

Then, our problem is to find u satisfying

$$\text{(P)} \begin{cases} \sigma_{ij,j} \equiv Au = 0 \quad \text{in} \quad \Omega \setminus \Gamma, \\ \sigma^+_{ij}\nu_j = \sigma^-_{ij}\nu_j = 0 \quad \text{on} \quad \Gamma^\pm, \\ u = 0 \quad \text{on} \quad \partial\Omega_-, \\ \sigma_{ij}\nu_j \equiv Tu = p \quad \text{on} \quad \partial\Omega_+. \end{cases}$$

Here and in what follows we use the summation convention. In the problem (P) $p = (p_1, p_2)^{\mathrm{T}}$ is a given vector of continuous functions on $\partial\Omega_+$ and $\nu = (\nu_1, \nu_2)^{\mathrm{T}}$ is the unit outward normal and

$$A \equiv \begin{pmatrix} \mu\triangle + (\lambda+\mu)\partial_1^2 & (\lambda+\mu)\partial_1\partial_2 \\ (\lambda+\mu)\partial_1\partial_2 & \mu\triangle + (\lambda+\mu)\partial_2^2 \end{pmatrix},$$

$$\triangle \equiv \partial_1^2 + \partial_2^2,$$

$$T \equiv \begin{pmatrix} (\lambda+2\mu)\nu_1\partial_1 + \mu\nu_2\partial_2 & \mu\nu_2\partial_1 + \lambda\nu_1\partial_2 \\ \lambda\nu_2\partial_1 + \mu\nu_1\partial_2 & \mu\nu_1\partial_1 + (\lambda+2\mu)\nu_2\partial_2 \end{pmatrix},$$

where λ and μ are *Lamé* constants satisfying that shearing strain $\mu > 0$, modulus of compression $3\lambda + 2\mu \ge 0$, in which case it is easy to see that the operator A is elliptic. And $\Gamma^\pm$ means both sides of Γ. Here for every $x \in \Gamma$, $\sigma^\pm_{ij}\nu_j = \sigma^\pm_{ij}(x)\nu_j(x)$ means the limit of $\sigma^\pm_{ij}(\bar{x})\nu_j(x)$ as $\bar{x} \in \Omega \setminus \Gamma$

tends to $x \in \Gamma$ along the normal $\nu(x)$. The limit values σ_{ij}^{+} and σ_{ij}^{-} may be different in general, therefore σ_{ij} may have a jump on Γ. At end-point of Γ (i. e. $(0,0)$) we assume

$$\lim_{x_1 \to 0} \sigma_{ij}\nu_j|_{x \in \Gamma^{\pm} \setminus \{(0,0)\}} = 0.$$

We introduce the class $\mathcal{K}$ of functions $u(x)$ with the properties (cf. [8]):

1. $u \in C^0(\overline{\Omega \setminus \Gamma}) \cap C^2(\Omega \setminus \Gamma)$,
2. $\nabla u \in C^0(\overline{\Omega \setminus \Gamma} \setminus \{(0,0)\})$,
3. in the neighborhood of $(0,0)$ there exist positive constant C and $\epsilon > -1$ such that

$$| \nabla u(x) | \leq C \, | x |^{\epsilon} \quad \text{as} \quad x \to 0,$$

4. for every $x \in \partial\Omega_{\pm}$ there exists a uniform limit of $\nabla_{\bar{x}} u(\bar{x})$ as $\bar{x} \in \Omega \setminus \Gamma$ tends to $x \in \partial\Omega_{\pm}$ along the normal ν_x.

Furthermore, the class $\mathcal{U}$ is defined by

$$\mathcal{U} = \{u \mid u \to 0 \text{ as } | x | \to \infty\}$$

and

$$C_{\gamma}^{0,\alpha} = \{f(x) \in C^{0,\alpha} \mid f(x) = O(| x |^{-\gamma}) \text{ as } | x | \to \infty\} \quad (1 < \gamma).$$

The usage of the plane elastic single and double layer potentials reduces the problem (P) to a system of singular integral equations. It is shown that this system is uniquely solvable in the appropriate H ö lder spaces by the Fredholm alternative (cf. [2, 10, 11]).

Theorem. *The problem* (P) *has a unique solution* $u \in \mathcal{K} \cap \mathcal{U}$ *for any* $p \in C_{\gamma}^{0,\alpha}(\partial\Omega_{+})$ *with any* $\alpha \in (0,1)$ *and any* $\gamma > 1$.

REFERENCES

1. *Chudinovich I. and Constanda C.* Existence and integral representations of weak solutions for elastic plates with cracks // J. Elasticity. 1999. V. 55, N 3. P. 169–191.
2. *Constanda C.* A Mathematical Analysis of Bending of Plates with Transverse Shear Deformation. Longman, Harlow, 1990.
3. *Constanda C.* The boundary integral equation method in plane elasticity // Proc. Amer. Math. Soc. 1995. V. 123, N 11. P. 3385–3396.
4. *Constanda C. (editor)* Integral Methods in Science and Engineering. Longman, Harlow, 1997.
5. *Fichera G.* Existence theorems in elasticity // Mechanics of Solids. 1984. V. 2, P. 347–389.
6. *Friedman A. and Liu Y.* Propagation of cracks in elastic media // Arch. Rational Mech. Anal. 1996. V. 136, N 3. P. 235–290.
7. *Itou H. and Tani A.* A boundary value problem for an infinite elastic strip with a semi-infinite crack // J. Elasticity. 2002. V. 66, N 3. P. 193–206.
8. *Krutitsukiĭ P. A.* The 2-D Neumann problem in a domain with cuts // Rendiconti di Mathematica. 1999. V. 19, N 1. P. 65–88.
9. *Landau L. D. and Lifshitz E. M.* Theory of Elasticity. Pergamon Press, 1986.
10. *Muskhelishvili N. I.* Singular Integral Equations. Noordhoff, Groningen, 1972.
11. *Vekua N. P.* Systems of Singular Integral Equations. Noordhoff, Groningen, 1967.