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# EQUIVARIANT HOMEOMORPHISMS IN CARNOT GROUPS AND SYMMETRIC SPACES AND THEIR QUASICONFORMALITY

**© Boris N. Apanasov**

apanasov@ou.edu

*University of Oklahoma, Norman, OK, USA*
*Sobolev Institute of Mathematics, Novosibirsk, Russia*

**1. Introduction.** In order to study homeomorphisms $M \to M'$ of two locally symmetric manifolds (or orbifolds) $M$ and $M'$ (locally modelled on a non-positive curved symmetric space $X$ or its Carnot group $\mathcal{N} = \partial X \backslash \{\infty\}$, one has equivariant homeomorphisms $f : X \to X$, that is homeomorphisms compatible with the action of the fundamental group $\Gamma = \pi_1(M) \subset \operatorname{Aut} X$ in the sense that $f \Gamma f^{-1} = \Gamma' \subset \operatorname{Aut} X$. For such homeomorphisms $f$ it is naturally to ask whether they are quasiconformal (or homotopic to quasiconformal ones). Since the classical works by M. A. Lavrentiev, L. V. Ahlfors, I. N. Vekua, P. P. Belinskii and Yu. G. Reshetnyak, such homeomorphisms were studied by many authors, especially in the case of Riemann surfaces, see [1, 2]. In particular, such quasiconformal homeomorphisms $f$ of a given domain $D \subset \mathbb{C}$ can be taken as generalized homeomorphic solutions $w = f(z)$ of Beltrami's equation $w_{\bar z} - \mu(z) w_z = 0$, where $w_z$ and $w_{\bar z}$ are locally square-integrable and $\mu(z)$ is a measurable function in $D$ with $||\mu||_\infty < 1$, see [1]. In the case of mappings of Riemann surfaces the coefficient $\mu(z)$ must also represent a Beltrami differential $\mu(z) d\bar z / dz$, i.e. this form must remain invariant under a change of the local parameter $z$ on the given Riemann surface, or to be $\Gamma$-invariant for the action of its fundamental group $\Gamma$ on $\mathbb{C}$, i. e. $\mu(\gamma(z)) \overline{\gamma'(z)} / \gamma'(z) = \mu(z)$ for all $\gamma \in \Gamma$, $z \in \mathbb{C}$, see [2, 3].

**2. Equivariant homeomorphisms in symmetric rank one spaces.** Especially such equivariant homeomorphisms become important for deformations of locally symmetric spaces of rank one, that is spaces modeled on $\mathbb{F}$-hyperbolic spaces $H^n_{\mathbb{F}}$ over numbers $\mathbb{F}$ that are either real $\mathbb{R}$, complex $\mathbb{C}$, quaternions $\mathbb{H}$, or Cayley numbers (octonions) $\mathbb{O}$. First, Mostow's theorem on rigidity of deformations $\rho : \Gamma \to H$ of lattices $\Gamma \subset H$ in the isometry group $H$ of a $\mathbb{F}$-hyperbolic spaces $H^n_{\mathbb{F}}$ implies that any homeomorphism $M \to M'$ of such locally symmetric spaces of finite volume is homotopy equivalent to an isometry between them. Then K.Corlette and M. Gromov – R. Schoen extended the G. A. Margulis superrigidity theorem for lattices $\Gamma$ in semisimple Lie groups $H$ of real rank at least two to the case of real rank one symmetric spaces which correspond to automorphisms groups $\mathrm{O}(n,1)$, $\mathrm{U}(n,1)$, $\mathrm{Sp}(n,1)$ and $\mathrm{F}_4^{-20}$ of real, complex and quaternionic hyperbolic spaces and the hyperbolic Cayley plane. Namely they proved that any lattice $\Gamma$ in $\mathrm{Sp}(n,1)$, $n \geq 2$, or $\mathrm{F}_4^{-20}$ is superrigid over archimedian fields and in $p$-adic case (which also implies its arithmeticity). For a geometric sense of such superrigidity for quaternionic manifolds, see [4, 5].

It is important to note that these quaternionic and octonionic hyperbolic spaces appear to be very rigid in the sense of quasiconformality. Namely, as P. Pansu [6] observed for the first time, any quasi-isometry there induces only "conformal" mapping at infinity, so one does not have any non-trivial quasiconformal homeomorphisms in the Carnot groups at their infinity. Nevertheless our constructions show, see [7–10]:

**Theorem 1.** *There is a big class of equivariant homeomorphisms in all type of symmetric spaces $H^n_{\mathbb{F}}$, corresponding to the so called "bending deformations"of locally symmetric rank one manifolds. In the real and complex hyperbolic spaces these bending homeomorphisms appear to be quasiconformal.*

**3. Quasiconformal instability in complex spaces.** In the remaining cases of real and complex hyperbolic spaces we have many non-arithmetic lattices and there are a number of constructions which show that superrigidity does not hold here either, see [7–10]. However there is a new type of rigidity for embeddings of uniform lattices $\Gamma \subset \mathrm{PU}(n-1,1) \hookrightarrow \mathrm{PU}(n,1)$ nearby their inclusions (here $\hookrightarrow$ is the natural lifting), see [11]:

**Theorem 2.** *Let $\Gamma \subset \mathrm{PU}(n-1,1) \hookrightarrow \mathrm{PU}(n,1)$ be a uniform lattice in $\mathrm{PU}(n-1,1)$, $n \geq 2$. Then for any its representation $\rho \operatorname{col} \Gamma \to \mathrm{PU}(n,1)$ nearby the inclusion, the group $\rho(\Gamma)$ preserves a complex totally geodesic $(n-1)$-subspace where, if $n > 2$, its action is conjugate to that of $\Gamma$.*

This new rigidity implies that one has no non-trivial quasiconformal homeomorphisms in $H^n_{\mathbb{C}}$ equivariant with respect to the action of mentioned lattices $\Gamma \subset \mathrm{PU}(n-1,1)$. We note however that both conditions of this rigidity, the action of a lattice in an *analytic* subspace and its co-compactness are essential. The latter follows from Theorem 1. The former is related to the existence of non-uniform lattices in complex hyperbolic geometry which appear to be quasiconformally instable, i. e. their small deformations induced by equivariant homeomorphisms of the complex hyperbolic space $H^n_{\mathbb{C}}$ cannot be induced by equivariant quasiconformal conjugations, see Apanasov [12, 13]:

**Theorem 3.** *Let a complex surface $M$ be the total space of a complex disc bundle over a non-compact (hyperbolic) Riemann surface $S = S_{g,k}$ of genus $g \geq 0$ with $k \geq 1$ punctures. Let us assume that $M$ has a complex hyperbolic structure $M_0 = (M, \rho_0)$ such that the surface $S$ is embedded in $M_0$ as its section, a totally geodesic complex 1-submanifold. Then the Teichmüller space $T(M)$ of complex hyperbolic structures on $M$ has a non-trivial smooth curve $\{M_t = (M, \rho_t), -\varepsilon < t < \varepsilon\}, \varepsilon > 0$, passing through $M_0$ and consisting of complex surfaces $M_t$ homeomorphic but not quasiconformally equivalent to $M_0$ for any $t \neq 0$.*

**4. Combinatorial conditions on quasiconformal conjugations in real spaces.** For the most flexible real hyperbolic geometry, we give a construction [14] which negatively answers the following question related to quasiconformal homeomorphisms equivariant with discrete Möbius actions in unit balls (or the question on the shape of quasiconformal balls in $\mathbb{R}^n$).

**Question 4.** *Whether any discrete Möbius group $G$ generated by finitely many reflections with respect to spheres $S^{n-1} \subset S^n$ and such that its fundamental polyhedron $P(G) \subset S^n$ is the union of two contractible polyhedra $P_1, P_2 \subset S^n$ of the same combinatorial type is quasiconformally conjugate in the sphere $S^n$ to some Fuchsian group $\Gamma$ preserving a round ball $B^n \subset S^n$?*

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