Международная конференция "Дифференциальные уравнения, теория функций и приложения", 2007, с. 493–494

УДК 517.95+519.46

ON COMMON ZEROES OF THE LAPLACE – BELTRAMI EIGENFUNCTIONS

© V. M. Gichev

gichev@ofim.oscsbras.ru

Omsk Branch of Sobolev Institute of Mathematics, Omsk, Russia

Let M be a compact connected closed orientable Riemannian C^∞-manifold, Δ be the Laplace – Beltrami operator on it, and

$$E_\lambda = \{u \in C^2(M) :\ \Delta u + \lambda u = 0\}.$$

be the eigenspace for an eigenvalue $-\lambda > 0$ (we assume that the functions are real valued). Let $H^p(M)$ denote de Rham cohomologies.

Theorem 1 ([1]). *Let M be as above.*

(1) *If $H^1(M) = 0$, then for any $\lambda \neq 0$ and every $u, v \in E_\lambda$ there exists $p \in M$ such that $u(p) = v(p) = 0$.*

(2) *If M is a homogeneous space of a compact Lie group G acting by isometries, then the converse is true: $H^1(M) \neq 0$ implies the existence of $\lambda \neq 0$ and a pair of $u, v \in E_\lambda$ without common zeroes.*

There is a simple example for the second assertion: let G be the circle group $\mathbb{T} = \mathbb{T}/2\pi\mathbb{Z}$, which acts on itself by the translations. Then $u(t) = cost$, $v(t) = sint$ have no common zero. In fact, (2) follows from this example and the existence of G-equivariant mapping $M \to \mathbb{T}$ for a nontrivial action of G on $\mathbb{T}$, which is a consequence of the assumption $H^1(M) \neq 0$.

For an eigenfunction u, $N_u = u^{-1}(0)$ is said to be the *nodal set*, and connected components of $M \setminus N_u$ are called *nodal domains.* The proof of (1) is based on the following properties of them:

(A) if U, V are nodal domains for $u, v \in E_\lambda$, respectively, and $V \subseteq U$, then $u = cv$ for some $c \in \mathbb{R}$;

(B) $u \in E_\lambda \setminus \{0\}$ cannot keep its sign near every point of N_u.

For a homogeneous space $M = G/H$ and any G-invariant Riemannian metric on M, each G-irreducible invariant subspace $E \subset L^2(M, \sigma)$, where σ is the invariant measure with the total mass 1 on M, is contained in some E_λ. For $a \in M$, let $\phi_a \in E$ be the function which realizes the evaluation functional at a: $\langle u, \phi_a\rangle = u(a)$ for all $u \in E$. For $a_1, \ldots, a_k, x, y \in M$, set $a = (a_1, \ldots a_k) \in M^k$, $\phi(x, y) = \langle \phi_x, \phi_y\rangle$, and

$$\Phi^a_{k,y}(x) = \det \begin{pmatrix} \phi(a_1,a_1) & \ldots & \phi(a_1,a_k) & \phi(a_1,y) \\ \vdots & \ddots & \vdots & \vdots \\ \phi(a_k,a_1) & \ldots & \phi(a_k,a_k) & \phi(a_k,y) \\ \phi(x,a_1) & \ldots & \phi(x,a_k) & \phi(x,y) \end{pmatrix}.$$

Further, let Φ_k be the mapping $(a, y) \to \Phi^a_{k,y}$, $n = \dim E - 1$, and set $U_n = \Phi_n(M^{n+1})$.

Theorem 2. *If $u \in E$, $u \neq 0$, then there exists a nontrivial continuous function $c(a, y)$ on $(N_u)^n \times M$ such that $\Phi^a_{n,y} = c(a, y)u$. Moreover, U_n is a compact symmetric neighbourhood of zero in E. For every $a \in M^n$, there exists a nontrivial nodal set which contains a; for a generic a, this set is unique.*

The construction, which is classical, can be applied to each finite dimensional space of continuous functions but usually the set U_n is small. The proof essentially uses (A).

Let M be the unit sphere $S^2 \subset \mathbb{R}^3$. Then $\lambda_n = n(n+1)$ is the n-th eigenvalue. The corresponding eigenspace $E_n = E_{\lambda_n}$ consists of spherical harmonics; they can be defined as restrictions to S^2 of homogeneous polynomials of degree n on $\mathbb{R}^3$ which are harmonic with respect to the ordinary Laplacian in $\mathbb{R}^3$. The zonal spherical functions ϕ_a can be written explicitly by the n-th Legendre polynomial; $\dim E_n = 2n+1$. Let $\nu(u,v)$ be the number of points in $N_u \cap N_v$, where $u, v \in E_n$. This set can be infinite. However, $\nu(u,v) \leq 2n^2$ for generic $u, v \in E_n$, and there are examples of u, v such that $\nu(u,v) = 2n^2$. The greatest lower bound is not known but partial results and computer experiments support the following conjecture: $\nu(u,v) \geq 2n$ for all $u, v \in E_n$. Also, there are examples for the equality (for all $n > 0$).

For problems and results (up to 2001) on the geometry of eigenfunctions, see the survey [2] and references in it.

REFERENCES

1. *Gichev V. M.* A note on common zeroes of Laplace–Beltrami eigenfunctions // Ann. of Global Anal. and Geom. 2004. V. 26. P. 201–208.
2. *Надирашвили Н., Тот Дж., Якобсон Д.* Геометрические свойства собственных функций// Успехи матем. наук. 2001. Т. 56, № 6. С. 67–88.