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BASIC BOUNDARY VALUE PROBLEMS FOR ANALYTIC FUNCTIONS IN A RING DOMAIN

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Four basic boundary value problems, namely, Schwarz, Dirichlet, Neumann, Robin for analytic functions or, equivalent, to the homogeneous Cauchy – Riemann equation, are considered in a concentric ring domain $R := \{z \in \mathbb{C}, r < |z| < 1\}$ of a complex plane $\mathbb{C}$, r is a real positive number.

Theorem 1. *The Schwarz problem for analytic functions in a ring domain* R

$$w_{\overline{z}} = 0 \text{ in } R, \quad Re\, w = \gamma \text{ on } \partial R, \ \frac{1}{2\pi i} \int\limits_{|z|=r} Im\, w(z) \frac{dz}{z} = c,$$

for $\gamma \in C(\partial R; \mathbb{R})$, $c \in \mathbb{R}$ *given, is uniquely solvable if and only if*

$$\frac{1}{2\pi i} \int\limits_{|z|=1} \gamma(z) \frac{dz}{z} = \frac{1}{2\pi i} \int\limits_{|z|=r} \gamma(z) \frac{dz}{z} = 0.$$

The solution is then given by

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \Big[\frac{\zeta + z}{\zeta - z} + 2 \sum_{n=1}^{\infty} \Big\{ \frac{r^{2n}\zeta}{r^{2n}\zeta - z} + \frac{r^{2n} z}{\zeta - r^{2n} z} \Big\} \Big] \frac{d\zeta}{\zeta} + ic.$$

Theorem 2. *The Dirichlet problem for analytic functions in a ring domain* R

$$w_{\overline{z}} = 0 \text{ in } R, \quad w = \gamma \text{ on } \partial R,$$

for $\gamma \in C(\partial R; \mathbb{R})$ *given, is solvable if and only if for* $z \in R$

$$\frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \frac{\overline{z} d\zeta}{1 - \overline{z}\zeta} = \frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \frac{\overline{z} d\zeta}{r^2 - \overline{z}\zeta} = 0.$$

Then the solution is unique and given by the Cauchy integral

$$w(z) = \frac{1}{2\pi i} \int\limits_{\partial R} \gamma(\zeta) \frac{d\zeta}{\zeta - z}.$$

Theorem 3. *The Neumann problem for analytic functions in a ring domain* R

$$w_{\overline{z}} = 0 \text{ in } R, \quad zw' = \gamma \text{ on } \partial R, \ w(z_{fix}) = c,$$

for $\gamma \in C(\partial R; \mathbb{R})$, $c \in \mathbb{C}$ *given,* $z_{fix} \in R$ *is solvable if and only if for* $z \in R$

$$\frac{1}{2\pi i} \int\limits_{|\zeta|=1} \gamma(\zeta) \frac{\overline{z} d\zeta}{1 - \overline{z}\zeta} = \frac{1}{2\pi i} \int\limits_{|\zeta|=r} \gamma(\zeta) \frac{\overline{z} d\zeta}{1 - \overline{z}\zeta} = 0,$$

$$\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta}=\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta}=0$$

are satisfied. If moreover $\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{d\zeta}{\zeta}=0$, *then the solution can be given by*

$$w(z)=c+\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\log\Big|\frac{1-z_{fix}\overline{\zeta}}{1-z\overline{\zeta}}\Big|^2\frac{d\zeta}{\zeta}-\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\log\Big|\frac{z_{fix}\overline{\zeta}-r^2}{z\overline{\zeta}-r^2}\Big|^2\frac{d\zeta}{\zeta}.$$

Theorem 4. *The Robin problem for analytic functions in a ring domain* R

$$w_{\overline{z}}=0 \text{ in } R,\quad w+zw'=\gamma \text{ on } \partial R,\quad z_{fix}w(z_{fix})=c$$

for $\gamma\in C(\partial R;\mathbb{R})$, $c\in\mathbb{C}$ *given,* $z_{fix}\in R$ *is solvable if and only if conditions*

$$\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{\overline{z}d\zeta}{1-\overline{z}\zeta}=\frac{1}{2\pi i}\int\limits_{\partial R}\gamma(\zeta)\frac{\overline{z}d\zeta}{r^2-\overline{z}\zeta}=0,\ \frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)d\zeta=0$$

are satisfied for any $z\in R$. *Then the unique solution is given by*

$$w(z)=c+\sum_{n=0}^{\infty}\frac{1}{n+1}\Big(\frac{1}{2\pi i}\int\limits_{|\zeta|=1}\gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}}\Big)(z^n-z_{fix}^{n+1})+$$

$$+\sum_{n=-\infty}^{-2}\frac{1}{n+1}\Big(\frac{1}{2\pi i}\int\limits_{|\zeta|=r}\gamma(\zeta)\frac{d\zeta}{\zeta^{n+1}}\Big)(z^n-z_{fix}^{n+1}).$$

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