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Variational method of solution of the inverse problem for elliptic equation

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The inverse problem of determining the unknown coefficient in the multidimensional elliptic equation of second order is considered, correctness of this problem is studied, existence and uniqueness theorems are proved. The regularizing numerical algorithm for the solution of the inverse problem is based on a variational approach.

Let D be bounded domain in the n-dimensional Euclidean space R^n, with sufficiently smooth boundary Γ. Assume that $x = (x_1, x_2, ..., x_n)$ is arbitrary point of D.

Consider the process describing by the following elliptic type equation

$$\mathcal{A}\psi + U(x)\psi = F(x), \quad x \in D, \tag{1}$$

where $\psi = \psi(x)$ is a solution of this equation, $U(x)$ is a unknown coefficient, $\mathcal{A}$ is the second order elliptic operator with given coefficients, $F(x)$ given function from $L_2(D)$. Let for this equation given the first and second boundary conditions

$$\psi|_\Gamma = g_0(x), \quad x \in \Gamma, \tag{2}$$

$$\left.\frac{\partial \psi}{\partial N}\right|_\Gamma = g_1(x), \quad x \in \Gamma, \tag{3}$$

where $g_0(x)$ and $g_1(x)$ are given functions from $W_2^1(D)$ and $L_2(D)$, N is the conormal of boundary Γ.

Our goal is to find the unknown function $U(x)$ and a function $\psi(x)$ from given up conditions(1)-(3).

Let

$$U_{ad} = \left\{ U = U(x) : \ U \subset L_p(D), \ \ U(x) \geq 0, \ \ \|U\|_{L_p(D)} \leq d, \ \ p \geq 2 \right\}$$

where $d > 0$ is a given number. The set U_{ad} is called the admissible set of unknown coefficients $U(x)$. Bounded in R^n functions $U(x)$, are belongs to the set U_{ad}. For every function $U(x)$ from U_{ad}, under solution of the first boundary value problem (1), (2), we shill understand a function $\psi_1(x)$ from $W_2^1(D)$, satisfying these conditions in generalizing sense. Analogically, solution of the second boundary value problem (1), (3), understudied the function $\psi_2(x)$ from $W_2^1(D)$, satisfying these conditions in generalizing sense.

Let us now consider the minimization problem for the functional

$$J_\alpha(U) = \|\psi_1(x) - \psi_2(x)\|^2_{L_2(D)} + \alpha \|U - U_0\|^2_{L_2(D)}, \tag{4}$$

in the set of admissible coefficients U_{ad}, where $\alpha \geq 0$ is parameter, $U_0 \in L_p(D)$ is the given function.

The inverse problem of determining the unknown potentials for equation (1) was studied in quantum mechanical scattering theory and spectral theory of differential operators in the definitions that the potentials are belongs to different class of the measurable functions. In this work inverse problem considered for coefficients belonging to set of admissible coefficients U_{ad}, which consists the Lebesgue integrating, generally, unbounded functions.

The problems (1),(2) and (1),(3) are the first and second boundary value problems, correspondingly, for the equation (1). Correctness of these problems have been studied in the series of works of O.Ladyzhenskaya,

J.Lions, etc. Solution of these problems are unique in class $W_2^1(D)$. However, special type a'priori estimates of solutions of these boundary value problems are established, which are necessary for studding of inverse problem (1)-(3). Solution of the variational problem (4), is called the generalizing solution of inverse problem (1)-(3)

Theorem 1 *The inverse problem (1)-(3) has a generalizing solution.*

Theorem 2 *There exists a dense in $L_p(D)$ a set V, such that for arbitrary $U_0 \in V$ and $\alpha > 0$ the inverse problem (1)-(3) has a unique generalizing solution.*

The adjoint problem for the variational problem (4), is the elliptic type and this problem has a solution, when exist generalizing solutions of the inverse problem (1)-(3). Due the adjoint problem respect to (1)-(3), studded the differentiability of the functional $J_\alpha(U)$ and founded the necessary condition for its minimum.

Theorem 3 *Let $p \leq \frac{8}{3}$. Then functional $J_\alpha(U)$ is Freshet differentiable on the set of admissible potentials U_{ad}.*

Theorem 4 *Functional $J_\alpha(U)$ is Freshet differentiable on the set of admissible coefficients U_{ad}.*

Necessary condition for the optimum of the functional (4) formulated in the variational inequality form throws of the gradient of this functional is.

Problem (1)-(4) is ill-posed. On the basis of the variational methods of minimization such as Newton and conditional gradient for the solution of the problem (1)-(3) the iterative procedure of regularization is constructed. Number α and step of iteration are parameters of regularization. Here discussed the stopping rules for choosing of these parameters.

Proving results generalized for arbitrary multidimensional elliptic equations and elliptic problems with unknown boundary.