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Direct and inverse problems for a model of electromagnetoelastic interactions

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1 Introduction

The theory of electromagnetoelasticity is concerned with the interacting effects of an externally applied electromagnetic field on the elastic deformations of a solid body. This theory is being fastly developed because of the possibilities of its extensive practical applications in diverse fields such as geophysics, optics, acoustic, damping of acoustic waves in the magnetic field and so on. For instance, while discussing the propagation of the seismic waves from the earth's mantle to its core, Cagniard [5] suggested that the existence of the earth's magnetic field may be taken into consideration for explaining certain phenomena concerning these waves. Knopoff [6] attempted to determine the effects of the magnetic field on the propagation of elastic waves on a geophysical scale. Though his conclusion that magnetic effects are very small may not be of much interest in seismic waves, his paper certainly gives an impetus to the theoretical development of this subject. For a more profound acquaintance with the modern state of the theory of electromagnetoelastic interactions the reader is referred to, e.g., Eringen and Maugin [4].

Following to Dunkin and Eringen [3] we can form the following system to describe electromagnetoelastic interactions in an electrically conducting elastic body subjected to a mechanical load

$$\begin{gathered}\sigma \mathbf{E} + \sigma\mu \frac{\partial \mathbf{U}}{\partial t} \times \mathbf{H} = \nabla \times \mathbf{H}, \\ \mu \frac{\partial \mathbf{H}}{\partial t} + \nabla \times \mathbf{E} = 0, \qquad \nabla \cdot (\mu \mathbf{H}) = 0,\end{gathered} \tag{1}$$

$$\rho \frac{\partial^2 \mathbf{U}}{\partial t^2} = \nabla \cdot \mathbf{T} + \mu \nabla \times \mathbf{H} \times \mathbf{H}, \tag{2}$$

where Hooke's law states that in an elastically isotropic solid the elastic stress tensor $\mathbf{T}$ is linearly related to the elastic strain tensor $\mathbf{S}$ according to the law

$$\mathbf{T} = \lambda\, tr\, \mathbf{S} \cdot I + 2\kappa\, \mathbf{S}. \tag{3}$$

Here I is the 3×3-identity matrix, and

$$\mathbf{S}:\ S_{ij} = \frac{1}{2}\Big(\frac{\partial U_i}{\partial x_j} + \frac{\partial U_j}{\partial x_i}\Big), \quad i, j = 1, 2, 3, \tag{4}$$

$\mathbf{U}$ is the displacement of the elastic body, $\mathbf{H}$ is the total (primary and induced) magnetic field, $\mathbf{E}$ is the electric field. The coefficients ϵ, μ are the electric permitivity and magnetic permeability, λ, κ are the Lamè elastic coefficients, correspondingly, and ρ is the density of the body. Let all the vectors be functions of $x_3 \equiv z$ and t only, and independent of the x_1 and x_2 coordinates. We assume that the displacement vector

$\mathbf{U}$ has the components $(0, 0, U_3)$. Under such assumptions for the case $\rho = \text{const}, \mu = \text{const}, \lambda = \lambda(z), \kappa = \kappa(z), \sigma = \sigma(z)$ we can form the following non-dimensional model system (cf. [1])

$$u_{tt} = (\nu^2 u_z)_z - p(h_1 h_{1z} + h_2 h_{2z}) + f, \quad h_{kt} = (r h_{kz})_z - (h_k u_t)_z - (r j_k)_z,\ k = 1, 2, \tag{5}$$

where h_1, h_2, u are dimensionless analogues of the functions H_1, H_2, U_3, $r^{-1} = \mu L V_0 \sigma$ is the magnetic Reynolds number, $p = \mu H_0^2 \rho^{-1} V_0^{-2}$, $v = \sqrt{(\lambda + 2\kappa)/\rho V_0^2}$ is dimensionless velocity of elastic waves propagation; and L, V_0, H_0 are characteristic values of length, velocity and magnetic field, respectively.

2 Direct problem

Consider the propagation of electromagnetoelastic waves in the presence of the external elastic and electromagnetic sources. Choosing the suitable orientation of these sources we can form in Q_T the following equations

$$u_{tt} = (\nu^2 u_z)_z - p(h_1 h_{1z} + h_2 h_{2z}) + f, \quad h_{kt} = (r h_{kz})_z - (h_k u_t)_z - (r j_k)_z,\ k = 1, 2, \tag{6}$$

where $r(z), \nu(z)$ are positive piecewise smooth functions and $f(z,t), j_k(z,t), k = 1, 2$, are piecewise smooth functions (dimensionless analogues of the electromagnetic and elastic sources), discontinuous at the points $z = z_m, m = 1, 2, \ldots, n$, $-l < z_1 < z_2 < \cdots < z_n < l$; p is a positive number.

The following first initial boundary-value problem is considered for equations (6) with the initial data

$$u(z, 0) = u_0(z),\ u_t(z, 0) = u_1(z),\ z \in \Omega, \quad h_k(z, 0) = h_{k0}(z),\ k = 1, 2,\ z \in \Omega, \tag{7}$$

and the boundary conditions

$$u(-l, t) = u(l, t) = 0,\ t \in (0, T), \quad h_k(-l, t) = h_k(l, t) = 0,\ k = 1, 2,\ t \in (0, T). \tag{8}$$

At the jump points $z_m, m = 1, 2, \ldots, n$, the following compatibility conditions are valid

$$[h_k] = [u] = 0, \quad [r(h_{kz} - j_k)] = \left[\nu^2 u_z\right] = 0, k = 1, 2\,. \tag{9}$$

The symbol $[v]$ denotes the jump of the function v as it passes through z_m.

2.1 Main results

Suppose that the functions $r, \nu, f, j_k, k = 1, 2$, the constant p and the initial data $u_0, u_1, h_{k0}, k = 1, 2$, in problem (6)–(9) enjoy the properties

(a) $r, \nu, f, j_k, k = 1, 2$, are supposed to be piecewise smooth functions with jumps at the points z_m : $-l < z_1 < z_2 < \cdots < z_m < l$; $0 < r_0 \leq r(z) \leq r_1 < \infty$, $0 < \nu_0 \leq \nu(z) \leq \nu_1 < \infty$ and p is a positive number;

(b) $h_{k0} \in C^\alpha(\overline{\Omega})$, $\alpha \in (0, 1), h_{k0}(\pm l) = 0, k = 1, 2$, and $u_0 \in \overset{o}{W}{}_2^1(\Omega)$, $u_1 \in L_2(\Omega)$.

Theorem 2.1 *If conditions* $(a) - (b)$ *are fulfilled, then problem (6)–(9) has a weak solution*

$$u(z, t) \in \overset{o}{W}{}_2^{1,1}(Q_T), \quad h_k(z, t) \in \overset{o}{V}_2(Q_T),\ k = 1, 2.$$

Theorem 2.2 *Problem (6)–(9) cannot have more then one weak solution.*

A weak solution of problem (6)–(9) is stable with respect to variations of the coefficients (except ν) and free terms of the equations, and also the initial conditions. This result is established in the case when ν is supposed to be a smooth enough function. Along with problem (6)–(9) consider the family of the problems

$$h^m_{kt} = (r^m h^m_{kz})_z - (h^m_k u^m_t)_z - (r^m j^m_k)_z, k = 1, 2, \ (z, t) \in Q_T, \tag{10}$$
$$u^m_{tt} = (\nu^2 u^m_z)_z - p(h^m_1 h^m_{1z} + h^m_2 h^m_{2z}) - f^m, \ (z, t) \in Q_T, \tag{11}$$
$$h^m_k(z, 0) = h^m_{k0}(z), \ k = 1, 2, \ z \in \Omega, \tag{12}$$
$$u^m(z, 0) = u^m_0(z), \ u^m_t(z, 0) = u^m_1(z), \ z \in \Omega, \tag{13}$$
$$u^m(\pm l, t) = h^m_k(\pm l, t) = 0, \ k = 1, 2, \ t \in (0, T), \tag{14}$$

where $m \in \mathbb{N}$. Suppose that $\nu(z), r^m(z), u^m_0(z), u^m_1(z), f^m(z, t), h^m_{k0}(z), j^m_k(z, t)$, $k = 1, 2$, are smooth functions satisfying the conditions

(c) $0 < \nu_0 \leq \nu(z) \leq \nu_1 < \infty$, $0 < r_0 \leq r^m(z) \leq r_1 < \infty$;

(d) $h^m_{k0} \in C^\alpha(\overline{\Omega})$, $\alpha \in (0, 1)$, $h^m_{k0}(\pm l) = 0$ and $u^m_0 \in \overset{o}{W}{}^1_2(\Omega)$, $u^m_1 \in L_2(\Omega)$,

and p is a positive number. In this case transmission conditions (9) can be dropped owing to the smoothness of the solution. Problems (10)–(14) have the unique weak solutions h^m_k, $k = 1, 2, u^m$, $m \in \mathbb{N}$.

Theorem 2.3 *Suppose the sequence $\{r^m\}$ is uniformly bounded and converges a.e. to r, while the sequences $\{f^m\}, \{u^m_0\}, \{u^m_1\}, \{j^m_k\}, \{h^m_{k0}\}$, $k = 1, 2$, converge to f, u_0, u_1, j_k, h_{k0}, $k = 1, 2$, in the norms of the spaces to which they belong according to the conditions of Theorem 2.1. Then in conditions (c)-(d) the weak solutions $u^m \in \overset{o}{W}{}^{1,1}_2(Q_T)$, $h^m_k \in \overset{o}{V}{}^{1,0}_2(Q_T)$, $k = 1, 2$ converge in such spaces to the weak solution u, h_k, $k = 1, 2$ of the limit problem (6)–(9).*

A proof of the results is based on the works [8, 9].

3 Inverse problem

Consider inverse problem of the determination of a set of the functions

$$h_k : \overline{Q}_T \to \mathcal{R}, k = 1, 2, \quad u : \overline{Q}_T \to \mathcal{R}, \quad \phi : [0, T] \to \mathcal{R}$$

satisfying equations (6)–(9) and additional information

$$\int_\Omega \eta(z) h_1 h_{1z} \, dz = -\frac{1}{2} \int_\Omega \eta_z h_1^2 \, dz = \psi(t), \quad t \in [0, T], \tag{15}$$

where $\eta \in \overset{o}{W}{}^1_2(\Omega)$,

$$\int_\Omega \eta(z) f(z, t) \, dz \geq \eta_0 > 0, \quad t \in [0, T],$$

and the functions $r, \nu, f, j_k, k = 1, 2$, the constant p and the initial data $u_0, u_1, h_{k0}, k = 1, 2$, satisfy conditions (a)-(b) in theorem 2.1.

3.1 Main result

For inverse problem (6)–(9), (15) the following existence and uniqueness theorem holds.

Theorem 3.1 *For sufficiently small values $T > 0$ inverse problem* (6)–(9), (15) *has a unique solution, which can be obtained by the method of successive approximations.*

A proof of the result is based on the work [7].

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