International Conference "Inverse and Ill-Posed Problems of Mathematical Physics",
dedicated to Professor M.M. Lavrent'ev on occasion of his 75-th birthday, August 20-25, 2007, Novosibirsk, Russia

Regularization of ill-posed problems with explicit data parametrization

V.K. Gorbunov*

* Ulyanovsk State University,
L. Tolstoy str., 42,
432970 Ulyanovsk, Russia
E-mail: vkgorbunov@mail.ru

The work was supported by the RFBR (grant 05-01-0001)

In our works [1, 2, 3, 4] (and others) the regularization methods for ill-posed problems with explicit parameterized input data are developing. Such a parametrization permits to state numeric problems that take into account the detailed data structure, and to obtain a solution of the data correction problem.

Here we consider an ill-posed minimization problem

$$M(y) = \operatorname{Argmin}\{f(x,y) : x \in P(y)\}. \tag{1}$$

in a Banach space $X \ni x$, with input data $y \in Y$ (metric space). The admissible set $P(y)$ is defined by the system

$$F(x,y) = 0, \quad g(x,y) \le 0, \quad x \in D, \tag{2}$$

where F and g are continuous mappings from $D \times Y$ to some Banach space $\mathcal{F}$ and to R^m correspondingly. The problem (1) is assumed to be resolved with exact date y^0, but the mapping $M(\cdot)$ is not unique and continuous defined on the bounded set of admissible data $Q_\delta(y^0) \subset Y$, which depends on the level of errors δ.

The more adequate regularization method for (1) is the transition to the Problem of Extended Minimization (PEM), that is the minimization of $f(x,y)$ on $x \in P(y)$ and on $y \in Q_\delta(\tilde{y})$, where $\tilde{y}$ is the given data from $Q_\delta(y^0)$ [2, 3, 4]. Here there is not any artificial regularization parameter, and we resolve simultaneously the data correction problem as well as the regularization of (1). The PEM can be understood as an expansion (on nonlinear problems) of the last Tikhonov's regularization concept [5] (for the system of linear equations) according to which an ill-posed computational problem should be replaced by a search of the element of minimal norm in the joint sets of solutions of the family of problems which are equivalent to the initial one with respect to accuracy of input data.

The PEM is a sufficient complicated problem, and in the case of regularity of $P(\cdot)$ we have offered and justified in [3, 4] simpler regularization of (1) via minimization of a Tikhonov's smoothing functional $T_{z\alpha}(x,y) = f(x,y) + \alpha\Omega(x-z)$ (where $\Omega(x)$ is a stabilizing functional, z is a trial point, and a small $\alpha > 0$) with respect to $x \in P(y)$.

Let $P(\cdot)$ is irregular, the system (2) can be decomposed on two systems

$$F_0(x,y) = 0, \quad g_0(x,y) \le 0, \tag{3}$$

and

$$F_1(x,y) = 0, \quad g_1(x,y) \le 0, \tag{4}$$

such that the first subsystem (3) defines a regular mapping $P_0(y) = \{x \in D : (3)\}$.

Introduce some penalty function $\psi(x,y)$ of complementary subsystem (4), and the penalty-smoothing functional $T_{z\alpha C}(x,y) = f(x,y) + \alpha\Omega(x-z) + C\psi(x,y)$ with a small $\alpha > 0$ and a big $C > 0$. We state the minimization problem

$$M_{z\alpha C}(y) = \operatorname{Argmin}\{T_{z\alpha C}(x,y) : x \in P_0(y)\}. \tag{5}$$

Also introduce the set of (z,Ω)-normal solutions of (1), i.e. $M_z(y) = \mathrm{Argmin}\{\Omega(x-z) : x \in M(y)\}$, the denotation of the deviation of a set A from a set B, i.e. $\beta(A,B) = \sup\{\inf\{\rho(a,b) : b \in B\} : a \in A\}$, and the notion of the *linearly boundedness from below* of the function $f(x,y)$ on the set $P(y)$ [4], that means existence of numbers $c_0(y)$ and $c_1(y)$ such that

$$x \in P(y) \Longrightarrow f(x,y) \geq c_0(y) + c_1(y)\sqrt{\Omega(x)}.$$

Theorem. *Let the problem (1) with exact data y^0 is being resolved, the admissible mapping P_0 of the problem (5) is regular and locally Lipschitzian on $Q_\delta(y^0)$, functions $f(x,y)$ and $\psi(x,y)$ are locally Lipschitzian, and function $f(x,y)$ is linearly bounded from below on $P(y)$ with any $y \in Y$. Then the problem (5) is compactly solvable with any $C > 0$ and $\alpha > 0$. If $C = C(\delta)$, $\alpha = \alpha(\delta)$, such that*

$$\lim_{\delta \downarrow 0} C(\delta) = \infty, \quad \lim_{\delta \downarrow 0} \left[\alpha(\delta) + \frac{\delta C(\delta)}{\alpha(\delta)}\right] = 0,$$

then

$$\lim_{\delta \downarrow 0} \sup\left\{\beta\left(M^{c(\delta)}_{z\alpha(\delta)}(y), M_z(y^0)\right) : y \in Q_\delta(y^0)\right\} = 0.$$

Also the new regularized variant of the relaxation penalty method [5] for the problem (1) with irregular constraints (6), as well as results of numerical experiments, will be presented.

We note the close work [7] on regularization of the minimization problem with irregular constraints.

References

1. V.K.Gorbunov On the regularization of extremum problems // Zhur. Vychisl. Matem. Matem. Fiz. 1991. Vol.31. No.2. P.235-248. 2. V.K.Gorbunov Regularization of compactly solvable problems // Moscow Univ. Comput. Math. Cyber. 1999. No.1. P.13-19.

3. V.K.Gorbunov Regularization of degenerate equations / inequalities under explicit data parameterization // J. of Inverse and Ill-Posed Problems (VSP). 2001. Vol.9. No 6. P.575-594.

4. V.K.Gorbunov Regularization of nonlinear ill-posed problems with parameterized data // Nonlinear analysis and nonlinear differential equations / Eds. V.A.Trenogin and A.Ph.Philippov. - Moscow: Fizmatlit. 2003. P.418-447 (in Russian).

5. A.N.Tikhonov On approximate systems of linear algebraic equations // Zhur. Vychisl. Matem. Matem. Fiz. 1980. Vol. 20. No.6. P.1373-1383.

6. V.K.Gorbunov The Relaxation Penalty Method and Degenerate Extremum Problems // Doklady Mathematics. 2001. Vol.63. No.2. P.219-223.

7. F.P.Vasil'ev The regularization methods for solving unstable minimization problems of the first kind with inexact given set // Zhur. Vychisl. Matem. Matem. Fiz. 2001. Vol.41. No.2. P.217-224.