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Inversion of the x-ray transform for 3D vector fields

S.G. Kazantsev*

*Institute of Mathematics,
Ak. Koptyug prosp., 4,
630090 Novosibirsk, Russia
E-mail: kazan@math.ru

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The new approach to the study of 3D vector tomography problems is presented. This approach can be generalized for the m-tensor tomography of arbitrary rank m.

Let us consider the Cartesian coordinate system $Ox_1x_2x_3$ on the 3D space $\mathbb{R}^3$ and let $\mathbf{x}$, $\boldsymbol{\xi} \in \mathbb{R}^3$, with $\mathbf{x} = (x_1, x_2, x_2)^T$, $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^T$. The scalar product of vectors $\mathbf{x}$, $\mathbf{y} \in \mathbb{R}^3$, with $\mathbf{x} = (x_1, x_2, x_3)^T$, $\mathbf{y} = (y_1, y_2, y_3)^T$, is denoted by $\mathbf{x} \cdot \mathbf{y} = x_1y_1 + x_2y_2 + x_3y_3$. By $\mathbb{B}^3 = \{\mathbf{x} : |\mathbf{x}| < 1\}$ and $\mathbb{S}^2 = \{\boldsymbol{\xi} : |\boldsymbol{\xi}| = 1\}$ we define the unit ball and the unit sphere in $\mathbb{R}^3$, respectively, where $|\mathbf{x}|$ and $|\boldsymbol{\xi}|$ are the usual Euclidian norms. We will always denote vectors and vector fields by boldface characters.

Let a vector field $\mathbf{a}(\mathbf{x}) = (a_1, a_2, a_3)^T$ is a vector function supported on the unit ball $\mathbb{B}^3$. The problem of vector tomography consists of reconstructing an unknown vector field when its x-ray transform is given.

Definition. *The vectorial (longitudinal) x-ray transform of vector field $\mathbf{a}(\mathbf{x})$ is the function $[\mathcal{D}\mathbf{a}](\boldsymbol{\xi}, \boldsymbol{\theta})$ defined on the direct product $\mathbb{S}^2 \times \mathbb{S}^2$ with the formula*

$$[\mathcal{D}\mathbf{a}](\boldsymbol{\xi}, \boldsymbol{\theta}) := \begin{cases} \int_{-2\boldsymbol{\xi}\cdot\boldsymbol{\theta}}^{0} \boldsymbol{\theta} \cdot \mathbf{a}(\boldsymbol{\xi} + p\boldsymbol{\theta})\mathrm{d}p & \text{if } \boldsymbol{\xi} \cdot \boldsymbol{\theta} \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Here the unit vector $\boldsymbol{\theta}$ is the direction along the integration line, which pass through the some $\boldsymbol{\xi} \in \mathbb{S}^2$.

We also say that $[\mathcal{D}\mathbf{a}](\boldsymbol{\xi}, \boldsymbol{\theta})$ is the vectorial x-ray transform in cone beam parametrization and $[\mathcal{D}\mathbf{a}](\boldsymbol{\xi}, \cdot)$ is sometimes called the cone-beam projection of $\mathbf{a}$.

It is known ([7]) that only the solenoidal (divergence-free) part of the vector field can be reconstructed from vectorial x-ray transform (1). So we will consider only solenoidal vector fields. A smooth vector field is called *solenoidal or divergence-free* if its divergence equals to zero,

$$\operatorname{div} \mathbf{a} = \operatorname{div} \begin{pmatrix} a_1 \\ a_2 \\ a_2 \end{pmatrix} := \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}.$$

The problem. *Our task is to solve the inverse problem $[\mathcal{D}\mathbf{a}](\boldsymbol{\xi}, \boldsymbol{\theta}) = f(\boldsymbol{\xi}, \boldsymbol{\theta})$, i.e. to find the solenoidal vector field $\mathbf{a}$ if its x-ray transform is given.*

We derive implicit inversion formulae for the transform (1) in two steps.

The first step. We proof that each solenoidal vector field $\mathbf{a}$ supported on the unit ball $\mathbb{B}^3$ can be represented uniquely as

$$\mathbf{a}(\mathbf{x}) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \sum_{n=0}^{\infty} \int_{\mathbb{S}^2} \left(u_1^{(n)}(\boldsymbol{\xi})\mathbf{e}_1(\boldsymbol{\xi}) + u_2^{(n)}(\boldsymbol{\xi})\mathbf{e}_2(\boldsymbol{\xi}) \right) C_n^{(3/2)}(\mathbf{x} \cdot \boldsymbol{\xi}) d\boldsymbol{\xi}, \tag{2}$$

where $C_n^{(3/2)}(\cdot)$ are the Gegenbauer polynomials of degree n and type $3/2$, the unit vector fields $\mathbf{e}_1(\xi)$ and

$\mathbf{e}_2(\xi)$ are generated by polar coordinates (φ, θ)

$$\mathbf{e}_1 := \frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\boldsymbol{\xi} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, \ \mathbf{e}_2 := \frac{\partial}{\partial\theta}\boldsymbol{\xi} = \begin{pmatrix} \cos\varphi\,\cos\theta \\ \sin\varphi\,\cos\theta \\ -\sin\theta \end{pmatrix}, \ \boldsymbol{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} = \begin{pmatrix} \cos\varphi\,\sin\theta \\ \sin\varphi\,\sin\theta \\ \cos\theta \end{pmatrix}.$$

Note, that vectors $\boldsymbol{\xi}$, $\mathbf{e}_1(\boldsymbol{\xi})$ and $\mathbf{e}_2(\boldsymbol{\xi})$ form the so called local moving triad, $\boldsymbol{\xi}\cdot\mathbf{e}_1(\boldsymbol{\xi}) = 0$, $\boldsymbol{\xi}\cdot\mathbf{e}_2(\boldsymbol{\xi}) = 0$ and hence $u_1^{(n)}(\boldsymbol{\xi})\mathbf{e}_1(\boldsymbol{\xi}) + u_2^{(n)}(\boldsymbol{\xi})\mathbf{e}_2(\boldsymbol{\xi})$ are the tangential vector fields on the sphere $\mathbb{S}^2$, $n = 0, 1, ...$.

The second step. We evaluate functions $u_1^{(n)}(\boldsymbol{\xi})$ and $u_2^{(n)}(\boldsymbol{\xi})$ in (2) by the known function $f(\boldsymbol{\xi}, \boldsymbol{\theta})$

$$u_1^{(n)}(\boldsymbol{\xi}) = \frac{2n+3}{16\pi^2}\int_{\mathbb{S}^2} \boldsymbol{\eta}\cdot\mathbf{e}_1(\boldsymbol{\xi}) C_n^{(3/2)}(\boldsymbol{\eta}\cdot\boldsymbol{\xi}) f(\boldsymbol{\eta}, \mathbf{e}_1(\boldsymbol{\xi}))\mathrm{d}\boldsymbol{\eta}, \tag{3}$$

$$u_2^{(n)}(\boldsymbol{\xi}) = \frac{2n+3}{16\pi^2}\int_{\mathbb{S}^2} \boldsymbol{\eta}\cdot\mathbf{e}_2(\boldsymbol{\xi}) C_n^{(3/2)}(\boldsymbol{\eta}\cdot\boldsymbol{\xi}) f(\boldsymbol{\eta}, \mathbf{e}_2(\boldsymbol{\xi}))\mathrm{d}\boldsymbol{\eta}. \tag{4}$$

Some information about the another reconstruction methods for 2D and 3D solenoidal vector fields can be find in [1, 2, 3, 4, 5, 6, 7, 8].

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