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Ill-posed Cauchy Problems in Hilbert and L_p Spaces

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The talk is concerned with deterministic and stochastic initial value problems for the abstract stochastic equation $X'(t) = AX(t) + B\mathbb{W}(t)$, $t \in [0,T]$, with operator A not generating a semigroup of class C_0 in a Hilbert space H, with H_1-valued white noise process $\mathbb{W}$, and $B \in L(H_1, H)$.

At the beginning we consider the problem with $B = 0$, that is the deterministic ill-posed Cauchy problem. The important example of such problem is the Cauchy problem for the backward heat conduction equation, whose regularization initially was considered by M.M. Lavrent'ev [1]. This example is an example of so called strongly ill-posed problem. If to restrict the equations to ones with differential operators $A = A(i\frac{\partial}{\partial x})$ and apply the Fourier transform to associated differential systems, then they may be classified under three types due to Gelfand–Shilov:

1. Systems correct in the Petrovsky sense: $\left\|e^{tA(\sigma)}\right\| \le C(1+|\sigma|)^h,\ t \ge 0,\ \sigma \in \mathbb{R}^n$,
2. Conditionally well-posed systems: $\left\|e^{tA(\sigma)}\right\| \le Ce^{at|\sigma|^h},\ a > 0,\ t \ge 0,\ \sigma \in \mathbb{R}^n,\ 0 < h < 1$,
3. All others

Using the modern semigroup techniques we constructed regularizing operators $\mathbf{R}_\varepsilon(t)$ as R-semigroups with operator $R = R(\varepsilon)$ depending on a regularizing parameter $\varepsilon \to 0$. These regularizing operators reduce the ill-posed differential Cauchy problem of the first and second types to the following well-posed boundary value problems, respectively:

$$\partial u(t,x)/\partial t = A(i\partial/\partial x)u(t,x), \quad x \in \mathbb{R}^n,\ t \in [0,T), \quad (\varepsilon\Delta - I)u(0,x) = f_\delta;$$

$$\partial v(t,x)/\partial t = \Delta v(t,x),\ 0 \le t \le \varepsilon, \quad v(0,x) = f_\delta,$$
$$\partial u(t,x)/\partial t = A(i\partial/\partial x)u(t,x), \quad x \in \mathbb{R}^n,\ t \in [0,T), \quad u(0,x) = v(\varepsilon,x),$$

We note that if to construct regularizing operators using the Fourier transform and not to reduce them to certain differential problems, then a regularizing operator may be obtained as a convolution with a δ–type sequence g_n: $\mathbf{R}_\varepsilon(t)f_\delta = G_t * g_n * f_\delta$, where $G_t(\cdot)$ is the inverse Fourier transform of the exponent $e^{tA(\cdot)}$. The equality reflects a common property of regularizing operators for ill–posed differential Cauchy problems to convolute G_t with a certain δ–type sequence g_n (where G_t is a kernel of the solution operator of the original problem). We used this property for constructing regularizing operators in L_p-spaces [2].

As for the stochastic problem, we have constructed solutions regularized in a broad sense [3], and as for regularization with a regularizing parameter, today there are more questions then answers, even in the setting of the problem, and they need to be answered.

References

[1] *Lavrent'ev M.M.* Conditionally well-posed problems for differential equations, Novosibirsk, Novos. Gos. Univ., 1973.

[2] *Melnikova I.V. and Vdovin M.V.* Regularization of ill-posed differential Cauchy problems in $L_p(\mathbb{R})$, $p \ge 1$, spaces. Proc. of 16-th Crimean Math.School KROMSH-2006, Simferopol, Taurida Nat.Univ., 2007.

[3] *Irina V. Melnikova and Alexei Filinkov* Abstract Cauchy Problem in spaces of stochastic distributions. Sovremennaja matematika. Fundamentalnie napravlenija. 2006, **16**, 96-109.