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Radiation-controlling boundary conditions for a problem of constrained evolution

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When specifying boundary conditions for evolution systems with differential constraints, the boundary data must be given compatibly with the constraint equations. In many cases, the compatibility condition is rather complex and involves both tangential and normal derivatives of the data at the boundary. This is particularly common for numerical general relativity, where methods were obtained to construct well-posed sets of boundary data that imply automatic constraint preservation (see [1, 3, 5, 6, 7, 8] and references therein). In these methods, the boundary conditions as a rule are chosen so as to guarantee the zero solution to the evolution equations satisfied by the constraint quantities. We propose a new approach, in which the auxiliary constrained evolution problem is used to construct sets of well-posed constraint-preserving data. The auxiliary system is obtained by adding combinations of the constraint equations to the evolution equations and has the property that it evolves the constraint quantities statically. The new approach enables us to formulate a well-posed initial boundary value problem for the constrained evolution system, construct new sets of radiation controlling constraint-preserving boundary conditions, and establish the well-posed energy estimates for both the constrained and free evolution problems.

We consider the problem of the vector wave equation

$$\partial_t^2 u_i = \partial^j \partial_j u_i, \tag{1}$$

subject to the differential constraint

$$\partial^i u_i = 0. \tag{2}$$

The choice of the model problem is motivated by re-formulations of Einstein's field equations (e.g., [4, 10, 9, 6]) in which second order in space equations are coupled to first order differential constraints. We use notations of general relativity, in which the repeated indices denote summation, small roman indices run from 1 to 3, the indices are raised and lowered with the flat metric $\delta_{ij} = \{1 \text{ if } i = j,\ 0 \text{ if } i \neq j\}$. Later we use bracketed indices to denote the anti-symmetric parts of an array, e.g., $w_{[ij]} = (w_{ij} - w_{ji})/2$.

By contracting both sides of (1) with ∂^i, commuting derivatives and introducing $C = \partial^i u_i$, we obtain

$$\partial_t^2 C = \partial^j \partial_j C. \tag{3}$$

Because of the dynamic form of (3), perturbations of the constraint are not localized at the boundary but propagate inside the domain. Since the constraint equation has to be satisfied at all times for a physical solution, propagating perturbations destroy the solution everywhere in a short time. The ideal boundary conditions, therefore, have to allow all perturbations of the constraint to leave the domain (with the smallest possible reflection) and block the outside perturbations from entering the domain. Data that is structured to ensure that $C \equiv 0$ for all times is called constraint-compatible, or *constraint-preserving*.

In this work we prove that (1), (2), taken as a *constrained evolution problem*, is related very closely to the equation

$$\partial_t^2 u_i = \partial^j \partial_j u_i - \partial_i \partial^j u_j. \tag{4}$$

It can be verified that for any solution of (4) there holds $\partial_t^2 C = 0$. That is, (4) evolves the constraint (2) statically. The following theorem can be obtained using the standard techniques of symmetric hyperbolic systems.

Theorem 1. Let n_i be the outward pointing unit normal to the boundary $\partial\Omega$. Denote by ∂_n the derivative in the direction n_i and by ∂_A the derivatives in the directions tangent to the boundary. Let, also u_n, u_A denote projections of u_i on the directions normal and tangential to the boundary, respectively. Let us assume that the initial data $u_i(0)$ and $\partial_t u_i(0)$ is given. Then there exists a unique solution to the problem (4) satisfying the initial data and boundary conditions

$$(\partial_t u_A + \partial_n u_A - \partial_A u_n) = \alpha_A^B(\partial_t u_B - \partial_n u_B + \partial_B u_n) + g_A, \tag{5}$$

where the smooth matrix α_A^B is defined on $\partial\Omega$, and satisfies $\|\alpha\| = \sup \frac{\|\alpha_A^B v_B\|}{\|v\|} \leq 1$. Moreover, the solution obeys the energy estimate:

$$\begin{aligned} &\sup_{0\leq t\leq T} [\|\partial_t u_i\|_{L_2(\Omega)} + 2\|\partial_{[j} u_{i]}\|_{L_2(\Omega)}] \\ &\qquad \leq c\int_0^T \|g_A\|_{H^{1/2}(\partial\Omega)} dt + [\|\partial_t u_i(0)\|_{L_2(\Omega)} + 2\|\partial_{[j} u_{i]}(0)\|_{L_2(\Omega)}]. \end{aligned} \tag{6}$$

The property of equation (4) to guarantee static evolution of the constraint quantity is used to establish the following corollary

Corollary 1. Let u_i be a solution to (4), (5). If the initial data satisfies the constraint (2), i.e., $C(0) = \partial^i u_i(0) = 0$ and $\partial_t C(0) = \partial^i \partial_t u_i(0) = 0$, then u_i satisfies (2) at any time.

By recalling the relationship between equations (1) and (4), and using Corollary 1 we prove the following theorem

Theorem 2. The constrained evolution problem (1), (2), provided with boundary conditions (5) is well-posed, i.e., for any initial data compatible with the constraint, a unique solution to (1), (2), (5) exists and satisfies the estimate (6).

It is often advantageous from the numerical point of view to solve the free evolution problem versus the fully constrained problem. We recall that only the evolution equation (1) is solved in the free evolution problem, while constraint (2) is monitored but not actively enforced on the solution. The following theorem gives an example of radiation-controlling boundary conditions that guarantee preservation of the constraint.

Theorem 3. Let $\alpha \in \mathbb{R}$, $|\alpha| \leq 1$, and the fields g and g_A defined on the boundary be compatible at corners (see [2] for the detail) and satisfy

$$\partial_t g = -\partial^A g_A, \quad g(0) = (1+\alpha)\partial_t u_n(0) + (1-\alpha)\partial_n u_n(0), \quad \text{on} \quad \partial\Omega. \tag{7}$$

Then there exists a unique solution to the free evolution problem (1) satisfying the constraint compatible initial data and the boundary conditions

$$\begin{aligned} (\partial_t u_A + \partial_n u_A - \partial_A u_n) &= \alpha(\partial_t u_A - \partial_n u_A + \partial_A u_n) + g_A, \\ (\partial_t u_n + \partial_n u_n) &= -\alpha(\partial_t u_n - \partial_n u_n) + g. \end{aligned} \tag{8}$$

Moreover, the solution satisfies the constraint equation (2).

Theorem 3 uses the assumption of the exact compatibility of data with the constraint equation. Such

assumption, however, is not practical in numerical applications where the equations are satisfied only approximately, and where one has to worry about propagation of small constraint violations. The next result generalizes Theorem 3 to the case of the arbitrary data.

Theorem 4. Let $\alpha \in \mathbb{R}$, $|\alpha| \leq 1$, and the fields g and g_A defined at the boundary be sufficiently smooth (see [2] for the detail). Let, in addition, the initial and boundary data be compatible, i.e.,

$$g(0) = (1+\alpha)\partial_t u_n(0) + (1-\alpha)\partial_n u_n(0), \quad \text{on} \quad \partial\Omega. \tag{9}$$

Then the unique solution $u_i \in L_2(\Omega)$, $\partial_{[j}u_{j]} \in L_2(\Omega)$, $\partial^l u_l \in L_2(\Omega)$ exists to (4) satisfying boundary conditions (8). Moreover, the solution satisfies the estimate

$$\begin{aligned}
&\sup_{0\leq t\leq T} [2\|\partial_t u_i\|_{L_2(\Omega)} + \|\partial_{[j}u_{i]}\|_{L_2(\Omega)} + \|\partial^l u_l\|_{L_2(\Omega)}] \\
&\quad \leq \; c\int_0^T \|g_A\|_{H^{1/2}(\partial\Omega)} dt + c_2 \mathrm{e}^T (c_1 \int_0^T \|\partial_t g + \partial^A g_A\|_{H^{1/2}(\partial\Omega)} dt \\
&\qquad + [\|\partial^l \partial_t u_l(0)\|_{L_2(\Omega)} + \|\partial^l u_l(0)\|_{L_2(\Omega)}]) + [2\|\partial_t u_i(0)\|_{L_2(\Omega)} + \|\partial_{[j}u_{i]}(0)\|_{L_2(\Omega)}].
\end{aligned}$$

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