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The variational spline method for singular differential equations

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The work is devoted to the explanation of the numerical Parametrization Method (PM) [1] for the variational problem that arises in connection with the initial problem for an implicit ODE

$$F(\dot{x}, x) = 0, \quad 0 \le t \le T, \quad x(0) = x^0,$$

where $x \in R^n$ and $F : R^{2n} \to R^n$, smooth transformation, in the case of arbitrary degeneracy of the Jacobi matrix $\partial F(\dot{x}(t), x(t))/\partial \dot{x}$ on the solution $x(t)$. The important particular case is the system of differential-algebraic equations (DAEs)

$$\dot{x}(t) = f(x(t), u(t)), \quad g(x(t), u(t)) = 0,$$

with conditions $x(0) = x^0, \quad u(0)) = u^0$. Here difficulties of numerical solving appear in the case of degeneracy of $\partial g(x(t), u(t))/\partial u$. Such singular problems are numerically ill-posed ones.

There are some special numerical methods for implicit ODEs and DAEs which have a finite index (the minimal number of differentiations that is sufficient for transformation of the system to the normal Cauchy form) up to 3 [2]. Note that Jacobi matrices mentioned above should have a constant rank here. However, not any system has a finite index and such a transformation can be rather complex and ill-posed operation when it is applied to functions defined with errors.

In view of the degeneracy we state the normal solution's problem defined by some trial function $z(t)$. The proposed numerical method is based on the minimization of the Tikhonov's stabilizing functional $\|F(\dot{x}, x)\|^2 + \alpha\|x - z\|^2$ and approximation of the solution $x_\alpha(t)$ by the "variational spline" with moving knots. Convergence of the spline to the normal solution is proved. The first and second derivatives of the discrepancy functional on the spline's parameters can be effectively calculated with the help of variational techniques and adjoint variables [3], [4]. Here we present also a simpler techniques for direct differentiation of the discrepancy with respect to the parameters [5]. Comparative analytical and numerical analysis of different variants of the PM will be presented.

References

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