International Conference «Inverse and Ill-Posed Problems of Mathematical Physics», dedicated to Professor M.M. Lavrent'ev in occasion of his 75-th birthday, August 20-25, 2007, Novosibirsk, Russia

Numerical methods for solving ill-posed problems with constraints and applications to inversion of the magnetic field

D.V. Lukyanenko*, Y. H. Pei **, A. G. Yagola*, Liu Gui-Rong****, N. A. Evdokimova*******

* Moscow State University,
Faculty of Physics,
Moscow, Russia 119992
E-mail: d_lukyanenko@yahoo.com

** DSO National Laboratories,
20 Science Park Drive,
Singapore 118230
E-mail: pyuanhan@dso.org.sg

*** Moscow State University,
Faculty of Physics,
Moscow, Russia 119992
E-mail: yagola@yahoo.com

****National University of Singapore,
Department of Mechanical Enginering,
Singapore 119260
E-mail: mpeliugr@nus.edu.sg

*****South Ural State University,
Chelyabinsk, Russia 454080
E-mail: naevdok@math.susu.ac.ru

The work of the authors was supported by the EERSS Program and the RFBR grant 05-01-00049

Abstract

Inversion of ill-posed problem from measurement data have been proposed use: i) Conjugate gradient projection method with regularization; ii) Conditional gradient method with regularization; iii) SVD with constraints with regularization method and iv) The method for solving two-dimensional integral equation of convolution type for vector functions using DFT method for Tikhonov functional. The performance of the proposed approach is demonstrated using simulated data with added noises.

Keywords: inverse problem, ill-posed problem, Tikhonov regularization.

1. Introduction

The main practical problem that investigated is the inversion of the magnetic field measured on the magnetometers in order to restore the magnetization vectors of a magnetic object.

Let us consider a magnetic object immersing in the earth magnetic fields. In the low magnetic fields, the linearity assumption can be made for ferromagnetic materials. The magnetization of the magnetic object can be simulated by the forward models, either numerical or analytic. In the forward magnetization model, the the induced magnetization M_i may be defined by the linear initial permeability of the ferromagnetic materials and the permanent magnetization M_p may be defined as coercive forces. Thus the total magnetization M is the linear combination $M = M_i + M_p$. In practice, there are errors in the measurement system, measurement data and magnetization models in terms of its location and dimension. The induced and permanent magnetization is differentiable using a proper procedure.

The permanent magnetization may be varied from one part to another over a magnetic object. In this inversion approach, our aim is to recover the complex permanent magnetization over the magnetic object. To represent the complicated magnetization, we simulate the magnetic object using numerical models, such as FEM. In this numerical model, we separate the magnetic object into a numbers of sub-volumes. The permanent magnetization inside each sub-volume can be varied from other sub-volumes either directions or amplitudes. The corresponding system of linear algebraic equations may be defined as,

$$[A][M] = [B] \qquad (1)$$

where $[B]$ is 3n column vector of measured magnetic fields. $[A]$ is a $3m \times 3n$ matrix which is the true forward model, derived by numerical models and geographic information, which generates the noise free data signal $[B]$. $[M]$ is $3m$ column vectors and m is the number of the sub-volumes. Since the vector B should be found from a measurement, so instead of B we have a vector B_δ such that $\| B_\delta - B\| \le \delta$, where $\delta>0$ is a measurement error.

The complex of programs for solving ill-posed problems has been developed by the following methods:

1. The conjugate gradient projection method with projection on set of vectors with nonnegative components so as regularization with the choice of the regularization parameter according to the generalized discrepancy principle, being based on a method described in the book [1], in two variants - with regularization algorithm and without, and also with an opportunity to use vector constraints.
2. The conditional gradient method so as regularization with the choice of the regularization parameter according to the generalized discrepancy principle, being based on a method described in [1], in two variants - with regularization and without, and also with an opportunity to use vector constraints $l_i \le x_i \le u_i$. Also for solving this problem was applied the conjugate gradient projection method with projection on set of vectors with nonnegative components so as regularization with the choice of the regularization parameter according to the generalized discrepancy principle.
3. The Singular Value Decomposition method (SVD), being based on a program for SVD a matrix from book [5] in two variants - without regularization and with added regularization algorithm by a principle of using a cut of smallest singular values in according with given error level.
4. The method for solving two-dimensional integral equation of convolution type for vector functions using DFT method for Tikhonov functional with the choice of the regularization parameter according to the generalized discrepancy principle, being based on a method described in the book [1].
5. The "SVD+constraints+regularization" method, being based on a method which at first time having applied regularization algorithm from the program for a SVD-method with regularization (point 3) and after this it apply the conjugate gradient projection method with constraints (point 1) to a finding of a solution of equation.

2. Method Implementations

<u>The conjugate gradient projection method</u>

For implementation first method was applied of *a conjugate gradient projection method for the solution is ill-posed problem in view on sets of special structure* according to strategy:

We will first treat the solution of the ill-posed problem

$$Az = u, \quad z \in M \subset Z = L_2, \quad u \in U, \tag{2}$$

under the condition that the operator A is continuous, linear and exactly known (h=0) from Z into U. We will assume that U is a Hilbert space. For the approached decision of a problem (2) in a case when M belongs to the limited, closed, convex set in Z, it is possible to accept any element z_δ for which $\|Az_\delta - u_\delta\| \le \delta$.

The functional $\Phi(z) = \|Az - u_\delta\|^2$ becomes the quadratic function $\varphi(z)$. Thus, we have passed to following problem: Construct a minimizing sequence for the functional $\Phi(z)$ on a convex set M. Note that any constraints can by written as

$$Fz \le g, \tag{3}$$

where F is a matrix of dimension $m_0 \times n$, m_0 being the amount of constraints defining the set, and g is a vector of length m_0. An inequality is to be understood as component wise inequalities.

If z is on the boundary of a set, then one or several inequalities in (3) may turn into an equality. We will call the set of indices for which at a point z the equality

$$\sum_{j=1}^{n} F_{ij} z_j = g_i \tag{4}$$

holds, the *set of active constraints* (at z), and we denote it by $I(z)$. By $\dim I$ we denote the number of element in I. The *matrix of active elements* is the matrix F_I of dimension $\dim I \times n$ whose rows of F that have row-number in $I(z)$.

We write the function $\varphi(z)$ as

$$\varphi(z) = (z, Qz) + (d, z) + e. \tag{5}$$

The simplest way to state the method of projection of conjugate gradients for minimizing the function (5) under the constraints (4) is in algorithmic form described in [1].

The conditional gradient method

For implementation of the second method was applied the conditional gradient method for solving problems on special sets according to strategy:

We will first treat the solution of the ill-posed problem (2). Consider the functional,

$$\Phi(z) = \left\| Az - u_\delta \right\|^2, \tag{6}$$

which is defined for all z, belong to the given set. Since A is linear, $\Phi(z)$ is a quadratic function of z.

In our case the problem of finding an approximate solution can be solved by minimizing $\Phi(z)$ on a set M. Here is not necessity to find the minimum $\Phi(z)$ on these set, but it suffices to find an element z_δ in these sets such that $\Phi(z_\delta) \le \delta^2$. Thus, to find an approximate solution of (6) we have to study the construction of a sequence minimizing some convex differentiable functional on a closed convex bounded set in a Hilbert space.

If, as in our case, the Frechet derivative of the functional satisfies a Lipchitz condition, then to solve the above p e.g., the conditional gradient method.

In the conditional gradient method one constructs, next to minimizing sequence $z^{(k)}$, an auxiliary sequence $\bar{z}^{(k)}$ as follows. Start with an arbitrary admissible point $z^{(0)} \in M$. Suppose $z^{(k)}$ has been constructed. Then $\bar{z}^{(k)}$ is a solution of the problem

$$\left(\Phi'\left(z^{(k)}\right), \bar{z}^{(k)}\right) = \min_{z \in M}\left(\Phi'\left(z^{(k)}\right), z\right). \tag{7}$$

This problem is solvable, i.e. there is a (in general, nonunique) point $\bar{z}^{(k)}$ at which the linear functional $\left(\Phi'\left(z^{(k)}\right), z\right)$ assumed its minimal value on M. Clearly, $\bar{z}^{(k)}$ belong to the boundary of M. Note that the problem (7) can be solved simply in case M is a bounded convex polyhedron in R^n. Then (7) is a linear programming problem, which can be solved by the simplex method, or if the vertices of M are known, by simply checking all vertices.

Suppose we have found $\bar{z}^{(k)}$. Then $z^{(k+1)}$ is constructed in accordance with

$$z^{(k+1)} = z^{(k)} + \lambda_k \left(\bar{z}^k - z^{(k)}\right), \tag{8}$$

where $\lambda_k \in [0,1]$ is the decision solution of the one-dimensional minimization problem

$$\Phi\left(z^{(k+1)}\right) = \Phi\left(z^{(k)} + \lambda_k\left(\bar{z}^k - z^{(k)}\right)\right) = \min_{\lambda \in [0,1]} \Phi\left(z^{(k)} + \lambda\left(\bar{z}^k - z^{(k)}\right)\right). \tag{9}$$

The latter problem comes down to minimizing $\Phi(z)$ on the segment $\left[z^{(k)}, \bar{z}^{(k)}\right]$. Since M is convex, $z^{(k+1)} \in M$. Thus, starting the iteration process with $z^{(0)} \in M$, the minimization process does not lead outside the boundary of M.

If $\Phi(z)$ is a quadratic function (this is true if A is linear), then (9) is a trivial problem: find the minimum of the parabola (with respect to λ on section [0,1]). If A is linear, the sequence $z^{(k)}$ thus constructed is minimizing for $\Phi(z)$ on M.

Thus, since in our case $\Phi(z)$ is a quadratic functional and (9) can be trivially solved, to construct efficient algorithms for the approximate solution of the ill-posed problem (2) on set M it suffices to study the efficient solution of the problem (7). The problem (7) can now simply be solved by checking the vertices of these polyhedron.

<u>The Singular Value Decomposition method (SVD)</u>

For implementation of the third and the fourth methods was applied a Singular Value Decomposition method (SVD) according to strategy described in [5].

<u>The method for solving two-dimensional integral equation of convolution type</u>

For implementation of the fourth method was applied a method for solving two-dimensional integral equation of convolution type for vector-function according to strategy:

Let's consider the two-dimensional integrated equation such as convolution for a vector-function:

$$A\vec{z} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} K(x-s, y-t)\ \vec{z}(s,t)\ ds\ dt = \vec{u}(x,y) \tag{10}$$

where we change designation: $t_1 = x,\quad t_2 = y,\quad s_1 = s,\quad s_2 = t$, in which the local supports of $\vec{z}(s,t)$ and $\vec{u}(x,y)$ lie in the interior of the rectangle $[0, 2r]\times[0, 2R]$, while outside their supports $\vec{z}(s,t)$ and $\vec{u}(x,y)$ put equal to zero on the whole rectangle.

We write down the finite-difference approximation of the functional $M^{\alpha}[z]$ for the equation:

$$\begin{aligned}\widehat{M}^{\alpha}[z] = &\sum_{k=0}^{n_1-1}\sum_{l=0}^{n_2-1}\sum_{i=1}^{3}\Big(\sum_{p=0}^{n_1-1}\sum_{j=0}^{n_2-1}\sum_{m=1}^{3} K_{k-p,l-j}^{im}\, z_{pj}^{m}\,\Delta x\Delta y - u_{kl}^{i}\Big)^2 \Delta x\Delta y + \\ &+ \alpha\sum_{k=0}^{n_1-1}\sum_{l=0}^{n_2-1}\sum_{i=1}^{3}\left\{ {z_{kl}^{i}}^{2} + \left[\frac{\partial^2 z^i(s_k,t_l)}{\partial s^2}\right]^2 + 2\left[\frac{\partial^2 z^i(s_k,t_l)}{\partial s\partial t}\right]^2 + \left[\frac{\partial^2 z^i(s_k,t_l)}{\partial t^2}\right]^2\right\}\Delta x\Delta y\end{aligned} \tag{11}$$

As in the previous section we arrive at

$$\left(\frac{\partial^2 \tilde{z}^i(s_k,t_l)}{\partial s^2}\right) = -\omega_m^2 \tilde{z}_{mn}^{i},\ \left(\frac{\partial^2 \tilde{z}^i(s_k,t_l)}{\partial s^2}\right) = -\omega_m^2 \tilde{z}_{mn}^{i},\ \left(\frac{\partial^2 \tilde{z}^i(s_k,t_l)}{\partial s\partial t}\right) = -\omega_m \Omega_n \tilde{z}_{mn}^{i}, \tag{12}$$

$$\omega_m = m\Delta\omega;\ \ \Delta\omega = \frac{\pi}{r};\ \ \Omega_n = n\Delta\Omega;\ \ \Delta\Omega = \frac{\pi}{R};$$

while the functional $M^{\alpha}[z]$ can be written as

$$\widehat{M}^{\alpha}[z] = \frac{\Delta x\Delta y}{n_1 n_2}\sum_{m=0}^{n_1-1}\sum_{n=0}^{n_2-1}\sum_{i=1}^{3}\left\{\left|\sum_{p=1}^{3}\tilde{K}_{mn}^{ip}\,\tilde{z}_{mn}^{p}\Delta x\Delta y - \tilde{u}_{mn}^{i}\right|^2 + \alpha\left[1 + \left(\omega_m^2 + \Omega_n^2\right)^2\right]\left|\tilde{z}_{mn}^{i}\right|^2\right\}. \tag{13}$$

3. Results

The proposed methods and corresponding computer programs were tested using the data of simulated magnetization models. The test case is a thin-shell type object with steel sheet thickness of 0.8mm, and has the following dimensions: length=1500mm, height=250mm and width=250mm. In this study, the magnetic object has been divided into sub-volumes in the FEM model. The permeability is unified over the body of the object. The Coercive forces of each sub-volume have been defined to change from one to others either in amplitudes and directions. The induced magnetization is generated by the external magnetic field applied to the structure and the permeability of the model. In the test cases, we assume that the induced magnetization is precisely predicted by FEM model and is the known parameters. Therefore the task of the inversion is to invert the amplitudes and directions of coercive force of each sub-volume.

Two test results are shown in Figures 1 and Figure 2 respectively. Figure 1 shows the inverted magnetization parameters of a test case which contains 14 sub-volumes and 42 magnetization parameters and these inverted parameters have been also compared with reference parameters. Figure 2 shows the inverted magnetization parameters of a test case which contains 20 sub-volumes and 60 magnetization parameters for is implemented the conditional gradient method (left figure) and the conjugate gradient projection method (right figure).

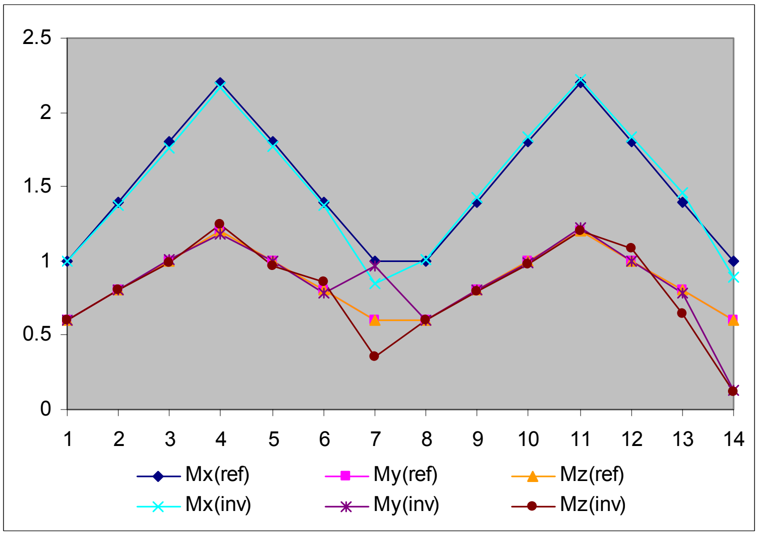
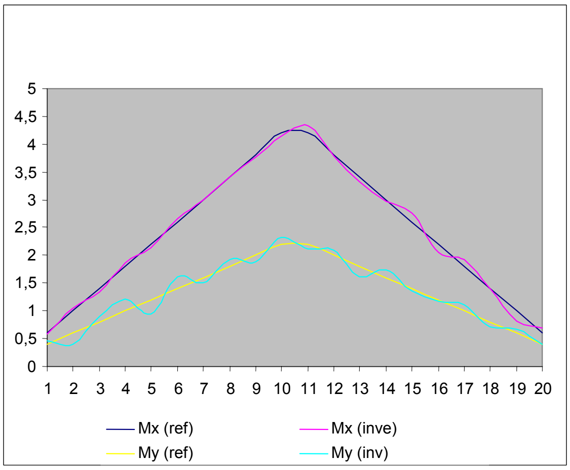


Fig. 1 Inversion results of 42 magnetization parameters



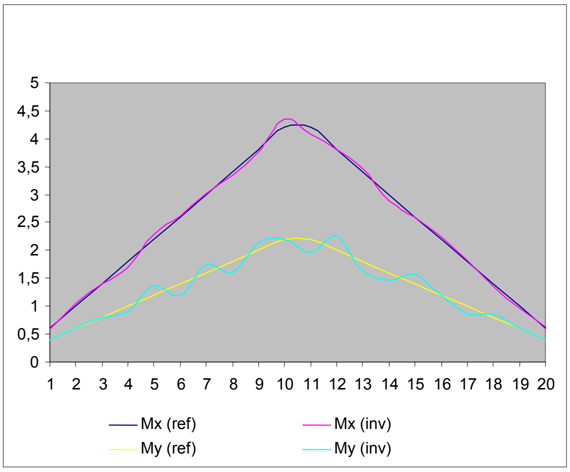


Fig. 2 Inversion results of 60 magnetization parameters for the conditional gradient method (left figure) and the conjugate gradient projection method (right figure)

4. Conclusion

The solutions of ill-posed problem of complex magnetization vector inversion have been proposed and tested. The performance of the proposed approach is demonstrated using simulated data.

References

1. A.N.Tikhonov, A.V.Goncharsky, V.V.Stepanov, A.G.Yagola. *Numerical methods for the solution of ill-posed problems*. Kluwer Academic Publishers, Dordrecht, 1995, p. 1-253.
2. Davis M. Himmelblau, Applied Nonlinear Programming, McGraw-Hill, 1972.
3. A. Bjorck, *Numerical Methods for Least SquaresProblems*, SIAM, Philadelphia, 1996.
4. J. D. Jackson, *Classical Electromagnetics*, New York, Wiley, 1975.
5. W.H. Press, S.A. Tcukolsky, W.T. Vetterling and E.P. Flannery, *Numerical Recipes in Fortran*, Cambridge University Press, 1992.