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Constraint Magnetization Parameter Inversion by Iterative Tikhonov Regularization

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Abstract

Recover magnetization vectors from magnetic measurements have many practical applications, especially in parameter identification. However, errors from forward models, sensors and measurement data can degrade the accuracy and quality of the parameter identification. In this paper, we present an approach of constraint magnetization parameter inversion based on iterative Tikhonov regularization. In this approach, we propose to add inequality constraints to the inversion problem based on expertises and model studies and to use an iterative procedure to adjust the values of constraints during the processes of inversions. An iterative truncated SVD Tikhonov regularization inversion process is constructed in the space of model parameters and the algorithm uses regularization function optimizes the inversion parameters based on the discrepancy and constraints criterion. The performance of the proposed approach is demonstrated using simulated data with added noises.

Keywords: ill-posed inverse problem, magnetization inversion, Tikhonov regularization

1. Introduction

Recovery of the magnetization vectors from magnetic measurement data has many practical applications, especially in source parameter identification [4]. Despite significant progress, inversion of magnetic measurement data still has many practical difficulties if the model and measurements are both subjected to noise and uncertainties [1]-[3]. Since the inverse problem is ill-posed, the major difficulty for such inversion problem is the theoretical non-uniqueness of the inversion, i.e., the results of inversions may be either inaccurate or the inversion trapped into local minima [5], [6].

In a practical problem, measurements are usually mixed with strong noises and uncertainties either from the measurement procedures, sensors or targets. The true parameters may be never known or they can only be estimated by the best knowledge of expertise. To reduce the non-uniqueness of the inverse solutions, *priori* knowledge to the magnetization, such as the directions or maximum vector strengths may be added to the inversion procedures. Therefore the expected parameter inversions are to recover the magnetization parameters with minimum inversion errors and obey the constraints. To solve the ill-posed problem, it is common to use regularization methods in order to limit the space-parameter zone and stabilize numerical solution.

In this paper, we propose use an iterative truncated SVD Tikhonov regularization inversion process to solve a type of discrete magnetization inversion problem. We propose to add inequality constraints to the inversion

problem based on expertises and model studies and to use an iterative procedure to adjust the values of constraints during the processes of inversions. In this approach regularization function is used to optimize the inversion parameters based on the discrepancy and constraints criterion.

2. Forward magnetization models

Since there is no general analytical model to describe the magnetostatic field of a compact ferrous object of arbitrary shape, we choose the homogeneous ellipsoid placed in a uniformity magnetic field. For a more complicated problem, numerical models such as FEM, may have to be used as well. The problem of a ferromagnetic ellipsoid in a uniform external magnetic field is solved use of ellipsoidal coordinates. If a ferromagnetic body is placed in a free space where $\mu = \mu_0$ and external magnetic field H_0, the permeability and geometry of the ferromagnetic object will result a demagnetization factor *N*. The magnetic field H is expressed as [7],

$$H = -\nabla\left[\frac{abc}{2}\vec{M}\cdot(xN^x\vec{i} + yN^y\vec{j} + zN^z\vec{k})\right] \tag{1}$$

where, $N^x + N^y + N^z = 1$ is the demagnetization factor of *x*, *y* and *z* axis and *a*, *b*, *c*, is the ellipsoid dimensions. For a rotationally symmetric prolate spheroid, the demagnetization factors are derived analytically.

To simulate more complex magnetization using analytical models, magnetic fields of current filament in 3D using the Biot–Savart law may be also sued. In such a method, the magnetic induction $\vec{B}$ is attributed to the current density distribution $\vec{J} = \nabla \times \vec{M}$. The magnetic field $\Delta\vec{B}$ at an arbitrary point due to the current-carrying filament with the current magnitude of ΔI is expressed by the Biot–Savart law as [7],

$$\Delta\vec{B}(\vec{r}) = \frac{\mu_0\,\Delta I}{4\pi\,r_0}(\cos\alpha_1 - \cos\alpha_2)\,\vec{n} \tag{2}$$

where α_1 and α_2 is the angle between terminal points of current filament with the calculation point, r_0 is the shortest distance between the extension line of current filament and the calculation point, $\vec{n}$ is the direction of magnetic field $\Delta\vec{B}$.

3. Inversion by iterative Tikhonov regularization with added constraints

In a magnetostatic inversion problem, we may define a magnetization source $\vec{M}$ which produces the magnetic field $\vec{B}$. If $\vec{M}$ exists and is unique, the problem can be defined as,

$$A\vec{M} = \vec{B}, \quad A \in R^{m\times n}, \quad \vec{B} \in R^m \tag{3}$$

The matrix *A* is the true forward model, derived by analytical models of equation (1) and (2), or numerical models, and geographic information, which generates the noise free magnetic fields $\vec{B}$. $\vec{B}$ is the measurement data with *m* entries and $\vec{M}$ is the magnetization vectors with *n* entries respectively. $n = 3\times m_e + m_c$, each ellipsoid magnetic model contains the 3-axis vectors of magnetization $3\times m_e$, $\vec{m}_e = m_{ei}\vec{i} + m_{ej}\vec{j} + m_{ek}\vec{k}$, and one component current filament contents magnetization m_c.

In practice, there are errors in the measurement system, measurement data, magnetization models in terms of its location and dimension. Incorporating the errors into (3) yields,

$$\left(A+\Delta A\right)\vec{M} = \left(\vec{B}+\Delta\vec{B}\right) \tag{4}$$

with ΔA and $\Delta\vec{B}$ denotes the error of model and measurement data respectively.

A direct application of the least squares solution, which is to solve the optimization problem,

$$\min_{\vec{M}} \quad f_0\left(\vec{M}\right)=\frac{1}{2}\left\| \left(A+\Delta A\right)\vec{M} - \left(\vec{B}+\Delta\vec{B}\right)\right\|^2 \tag{5}$$

Using the least squares norm of the data fitting term, typically runs into trouble. The solution system (4) is overdetermined in our problem. It is well-known that the inverse problem is ill-posed. When problem is ill-posed, that is when existence of the solution with respect to measurement data is not verified, it is common to use regularization methods in order to limit the space-parameter zone. Most common used methods are Tikhonov's, Levenberg's and Levenberg-Marquardt's [4]-[6].

Tikhonov regularization is perhaps the most common regularization scheme. To solve equation (5) with Tikhonov regularization, for a stable numerical solution of the equation (5), we form a sum of data misfit term $\left\| A\vec{M}-\vec{B}\right\|^2$ and a regularization term $\alpha^2\left\|\rho\left(\vec{M}-\vec{M}_0\right)\right\|^2$ with a weighted matrix ρ**,** and find the solution based on minimization of the Tikhonov functional [8],

$$f_\alpha(\vec{M}) = \left\| A\vec{M}-\vec{B}\right\|^2 + \alpha^2\left\|\rho\left(\vec{M}-\vec{M}_0\right)\right\|^2 \tag{6}$$

where $\alpha > 0$ is the regularization parameter that should be chosen based on the discrepancy principle. $\vec{M}_0$ is an expected solution of $\vec{M}$ and ρ is a weighted matrix which is set to an identity matrix in our proposed approach.

If the system is linear as in (6), An explicit solution, denoted by $\hat{M}$, is given by

$$\hat{M} = \vec{M}_0 + \left(A^T A + \alpha^2 I\right)^{-1} A^T\left(\vec{B} - A\vec{M}_0\right) \tag{7}$$

where I is a $m\times n$ identity matrix. For $\alpha = 0$ this reduces to the least squares solution of an overdetermined problem ($m > n$).

By applying singular value decomposition, $A = UWV^T, U \in R^{mxn}, V \in R^{nxn}, W \in R^{nxn}$, where U and V is the left and right singular vector, respectively, W with elements positive or equal to zero is the singular value.

Since the matrix condition number $\kappa(A) = \sigma_{\max}(A)/\sigma_{\min}(A)$ associated with the linear equation $A\vec{M} = \vec{B}$ gives a bound on how inaccurate the solution $\vec{M}$ will be after approximate solution, the smaller singular values of matrix A is therefore discarded based on minimizing discrepancy principle [9][10]. The truncated SVD is no longer an exact decomposition of the original matrix *A*, but it improves the matrix condition number by truncations of smaller singular values, which is in a useful sense to reject uncertainties of models and errors of measurements. The problem (6) can be solved by the truncated singular value decomposition as,

$$f_\alpha(\vec{M}) = \left\| A\vec{M}-\vec{B}\right\|^2 + \alpha^2\left\|\rho\left(\vec{M}-\vec{M}_0\right)\right\|^2 = \left\| W\cdot V^T\vec{M} - U^T\vec{B}\right\|^2 + \alpha^2\left\|\rho\left(\vec{M}-\vec{M}_0\right)\right\|^2 \tag{8}$$

If the expected solution $\vec{M}_0$ can be determined by *priori* knowledge to the magnetization, either measurements or expertise, such as the directions or maximum vector strengths, solution of equation (8) will

toward the true values of $\vec{M}$. In practical cases, we are usually difficult to determine the $\vec{M}_0$ precisely but we perhaps be able to determine its range.

Note that for an ill-posed problem one must necessarily introduce some additional assumptions in order to get a stable solution. Certainly, *a priori* knowledge of the range of the solution $\vec{M}$, *i.e.*, the constraints, takes great advantage to stabilize the solution to the ill-posed problem. If we define the expected solution $\vec{M}_0$ as the inequality constraints as,

$$\begin{cases} f_\alpha(\vec{M}) = \left\| W \cdot V^T \vec{M} - U^T \vec{B} \right\|^2 + \alpha^2 \left\| \rho\left(\vec{M} - \vec{M}_0\right) \right\|^2 \\ \vec{M}_{0i\min} \le \vec{M}_{0i} \le \vec{M}_{0i\max} \qquad (i = 1,2,3,\cdots,n) \end{cases} \tag{9}$$

where $\vec{M}_{0i\min}$ and $\vec{M}_{0i\max}$ represents the minimal and maximal limit values of inequality $\vec{M}_{0i}$ respectively. The initial values of inequality constraints can be formed based on expertises and model studies results.

It is clear that with a large regularization parameter α, we effectively ignore the measured data and push $f_\alpha\left(\vec{M}\right)$ towards the smooth solution. On the other hand, if α is small, $f_\alpha\left(\vec{M}\right)$ will close to the usual least-squares solution.

We shall describe how to choose regularization parameter α, to take advantage of this property for solving the constraint magnetization parameter inversion problem.

The proposed algorithm starts first by setting an initial value of regularization parameter α_{ini} and a calculation of equation (8) without any constraint for magnetization parameter inversion and finds an initial optimum regularization parameter α_i by discrepancy principle.

The proposed algorithm is then implemented by the following steps:

1) At iteration step *k*, change regularization parameter $\alpha_{\text{k+1}} = \alpha_{\text{k}} + f(\delta)\,\alpha_{\text{k}}$;
2) Evaluate constraints, if $\vec{M}_{0i\min} \le \vec{M}_i \le \vec{M}_{0i\max}$, $(i = 1,2,3,\cdots,n)$, record the k_{th} inversion solution;
3) If $\vec{M}_i \le \vec{M}_{0i\min}$ or $\vec{M}_i \ge \vec{M}_{0i\max}$, set $\vec{M}_i = \vec{M}_{0i\min}$ or $\vec{M}_i = \vec{M}_{i0\max}$. Solve the inversion problem to the magnetization parameters by taking out i_{th} magnetization vector $\vec{M}_i$ in equation (8) and evaluating constraints until all the inverted parameters obey the defined constraints. Record k_{th} inversion solutions;
4) Change regularization step length $f(\delta)$ based on $r = (\varepsilon_k - \varepsilon_{k-1})/(\alpha_k - \alpha_{k-1})$ so as to accelerate the convergence speed;
5) Exit criterion by comparing the inversion discrepancy of iteration *k* and *k*-1: $\varepsilon_k > \varepsilon_{k-1}$.

The proposed approach uses an iterative truncated SVD Tikhonov regularization inversion process in the solutions of discrete magnetization inversion problem. Regularization function is used to optimize the inversion parameters based on the discrepancy and constraints criterion.

4. Iterative determinations of added constraints

It is obvious that the better inversion results greatly depend on the right choices of the values of the inequality constraints. However, there is still a problem how to determine the values of inequality constraints in the real applications. In this paper, we propose an iteratively procedure to adjust the values of inequality constraints via observing the inverted parameters with comparisons of forward models, measurement data and error analysis. We found that it is a useful approach to improve the accuracy and stability of the solution of inversion problem.

The procedure may be illustrated by a test case, in which a simulated model, with 276 discrete magnetization vectors, is formed. The magnetic field data are presented by free of noise. Figure 1 shows the comparisons of inversion results with broad and severe inequality constraints. For the inversion with broad inequality constraints, the inversion results have wide range, which have quite large differences with the parameters of forward models, but with trivial inversion errors of the magnetic fields. For the inversion with severe inequality constraints, which determined based on forward models, the inversion results have narrow range parameters, which close to the parameters of forward models, with the similar level of trivial inversion errors of magnetic fields.

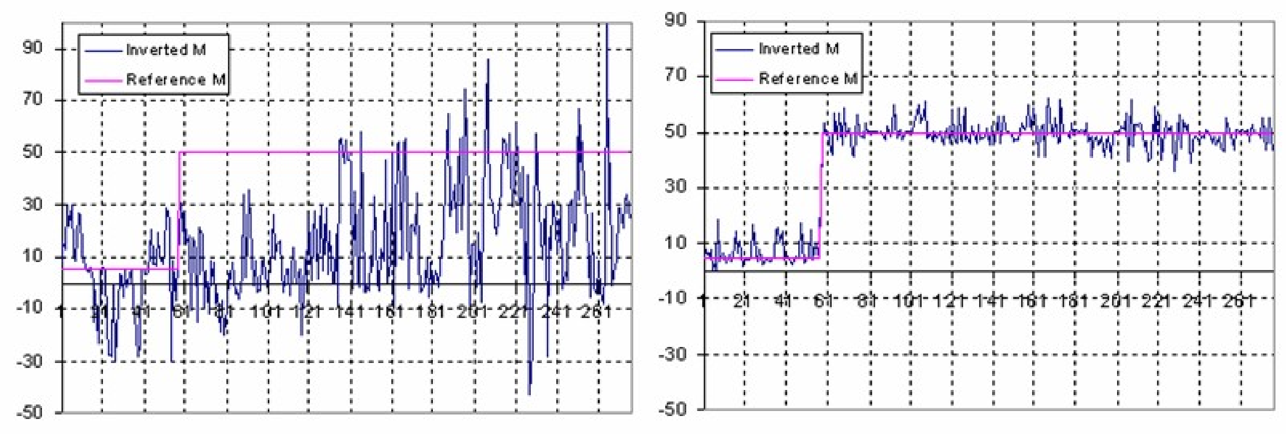

Figure 1 Inversion results with broad inequality constraints (left) and with severe inequality constraints (right)

5. Results

The proposed technique has been tested with simulated data. The inverse problem has been simulated use of both ellipsoid and current filament models. One of representative results is shown in Figures 2 and Figure 3. The test case contains 33 magnetization vectors and random noises have been added into the modelling data based on our experiences of real measurements to simulate the real scenarios. Figure 2 shows the magnetic fields below the simulated magnetization models. Figure 3 shows the inverted magnetization vectors and these vectors have been also compared with reference vectors.

The magnetization vectors, $\vec{m}_e = m_{ei}\vec{i} + m_{ej}\vec{j} + m_{ek}\vec{k}$, if it has relatively smaller strength, have been observed, from the inversion results, that usually have larger inversion errors. These inversion errors are shown in Figure 3. From these inversion results, the proposed inversion approach shows that it can recover the magnetization vectors with good accuracy. The accuracies are usually better than those unconstraint inversion approaches based on our cases studies.

6. Conclusion

This paper presents the solution of ill-posed discrete magnetization inversion problem. We have proposed an approach of iterative truncated SVD Tikhonov regularization inversion and to use regularization function to optimize the inversion parameters based on the discrepancy and constraints criterion. We propose to add inequality constraints based on expertises and model studies and to use an iterative procedure to adjust the values of constraints during the processes of inversions. The computations show that this approach may recover the magnetization vectors with better accuracy.

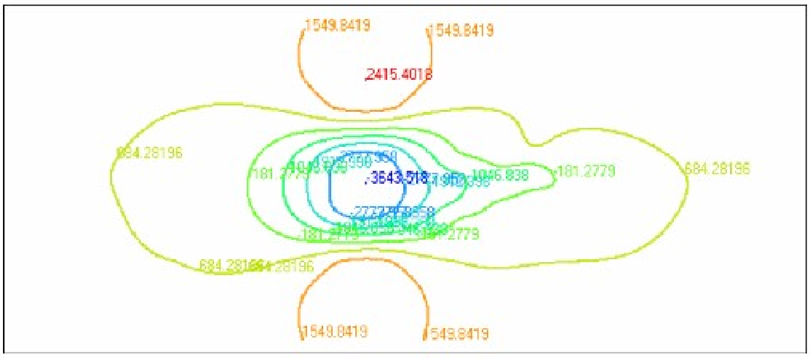

Figure 2 Magnetic field contours below the simulated magnetization models.

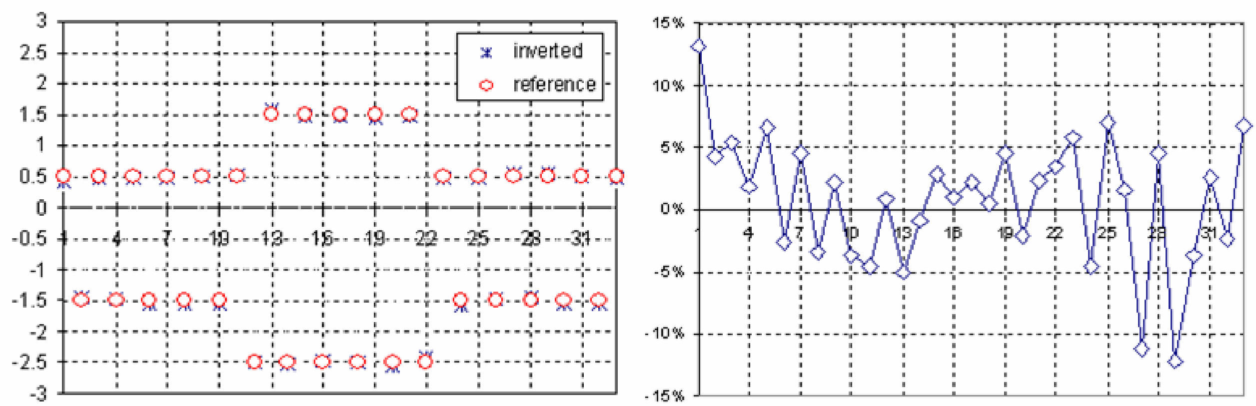

Figure 3 Inversion results and comparisons with reference vectors (left) and Inversion errors (right)

References

1. Ventouras, E.M.; Papageorgiou, C.C.; Uzunoglu, N.K.; Christodoulou, G.N., *Tikhonov regularization using a minimum-product criterion: Application to brain electrical tomography*, Proceedings of the 23rd Annual Conference, IEEE, Vol.1, Oct. 2001.

2. Colinas, J.; Goussard, Y.; Laurin, J.-J., *Application of the Tikhonov regularization technique to the equivalent magnetic currents near-field technique, Antennas and Propagation*, IEEE Trans. on Vol.52, Issue 11, Nov. 2004.

3. Xudong Chen; Kevin O'Neill; Tomasz M. Grzegorczyk; Jin Au Kong, *Spheroidal Mode Approach for the Characterization of Metallic Objects Using Electromagnetic Induction*, Geoscience and Remote Sensing, IEEE Trans. on, Vol.45, Issue 3, Mar.2007.

4. R. L. Parker. *Geophysical Inverse Theory*. Princeton University Press, Princeton NJ, 1994.

5. R. Fletcher, *Practical Methods of Optimization*, Chichester, UK, Wiley, 1980.

6. Davis M. Himmelblau, *Applied Nonlinear Programming*, McGraw-Hill, 1972.

7. J. D. Jackson, *Classical Electromagnetics*, New York, Wiley, 1975.

8. A.N.Tikhonov, A.V.Goncharsky, V.V.Stepanov, A.G.Yagola. *Numerical methods for the solution of ill-posed problems*, Kluwer Academic Publishers, Dordrecht, 1995.

9. Ake Bjorck, *Numerical Methods for Least Squares Problems*, SIAM, Philadelphia, 1996.

10. D.V. Lukyanenko, Y. H. Pei , A. G. Yagola, Liu Gui-Rong, N. A. Evdokimova, *Numerical methods for solving ill-posed problems with constraints and applications to inversion of the magnetic field*, International Conference Inverse and Ill-Posed Problems of Mathematical Physics, Aug. 2007, Russia.