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Estimating the gradient descent method convergence rate for the sideways heat conduction problem

A.V. Penenko*, **S.I. Kabanikhin****

* IM SB RAS,
Ak. Koptyug prosp., 4,
630090 Novosibirsk, Russia
E-mail: aleks@ommgp.sscc.ru

** IM SB RAS,
Ak. Koptyug prosp., 4,
630090 Novosibirsk, Russia
E-mail: kabanikh@math.nsc.ru

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Convergence issues arise if inverse problem is solved with an iterative method. In this work we present a study of convergence rate of the gradient method applied to one-dimensional sideways heat conduction equation (1)-(4):

$$u_t = u_{xx}, t \in [0,T], x \in [0,1], \qquad (1)$$
$$u(1,t) = f(t), t \in [0,T], \qquad (2)$$
$$u_x(1,t) = 0, t \in [0,T], \qquad (3)$$
$$u(x,0) = 0, x \in [0,1], \qquad (4)$$
$$u(0,t) = q(t), t \in [0,T], \qquad (5)$$

where f(t) is assumed to be known, whilst $q(t)$ is unknown but there is an *a priori* bound for it:

$$\|q(.)\|^2_{L_2(0,T)} < M.$$

To solve the problem, an output least square one is posed with introducing the functional:

$$J(q) = \frac{1}{2}\left\|u(1,.;q) - f(.)\right\|^2_{L_2(0,T)}, \qquad (6)$$

where $u(x,t;q)$ is the solution of the direct heat conduction problem (1),(3)-(5). The optimization problem obtained is solved with the gradient method. Following the framework proposed in [1] to estimate a convergence rate of the gradient based method we are to combine 3 facts:

- conditional stability estimate for the problem (1)-(4). In case of the sideways heat equation this one can be taken from [2] and be slightly modified for our needs;
- an estimate of the functional’s (6) decrease rate;
- the fact that the conditional stability estimate is applicable to the sequence, produced by the gradient-based method. This is a corollary to the statement, that a gradient-based method doesn’t increase the error $\|q_n - q^*\|_{L_2(0,T)}$, where q_n is the n-th approximation to the solution, q^* is such that $u(1,t;q^*) = f(t)$ [3].

As a result, the estimate of the following form can be obtained

$$\|u(x,.;q_n) - u(x,.;q^*)\|_{L_2(0,T)} \leq F(n, \|q_0 - q^*\|_{L_2(0,T)}, x, \|\mathrm{A}\|), \qquad (7)$$

where $F(n,w,x,r)$ is some function, q_0 is an initial guess to the solution and A is the linear operator specified by the formula:

$$A : q(t) \mapsto u(1,t;q).$$

These results can be generalized to the case of noisy data with an additional term involving noise level added to (7).

Theoretical results are compared with those of numerical runs.

References

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