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Tomographic image reconstruction using Neumann decomposition

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Abstract

In this study, we consider a new modification of the iterative algorithm of two-dimensional computed tomography using the Neumann expansion technique. Algorithm uses the convolution of Radon transform of unknown function with Shepp - Logan kernel in iterations. Our elaborations applies to a few - view tomography problem, and numerical simulations have shown that better reconstruction quality can be achieved by the new iterative algorithm compared to the Gerchberg - Papoulis algorithm. Application of this algorithm to the steganography problem is proposed.

Keywords: inverse problem, Neumann decomposition, Shepp - Logan iterative algorithm, Gerchberg - Papoulis algorithm, a few-view tomography, steganography.

1. Theory

The main integral transform of computed tomography is Radon transform. Using the Dirac delta function, it can be defined by [1]:

$$f(\xi, p) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x, y)\delta(-x \sin\xi + y\cos\xi - p)dxdy, \tag{1}$$

where $f(\xi, p)$ is a projection function or sinogram. To reconstruct function $g(x, y)$ from $f(\xi, p)$, the inverse Radon transform is applied to it. Analytical form of the inverse Radon transform is [1]:

$$g(x, y) = -\frac{1}{2\pi^2}\int_0^{\pi} d\xi \int_{-\infty}^{\infty} \frac{f(\xi, p)dp}{(p - p_0)^2},\ p_0 = -x\sin\xi + y\cos\xi. \tag{2}$$

There are several techniques by which the inverse Radon transform can be calculated, the most common is Filtered Back-Projection algorithm (FBP) [2].

Classical Shepp - Logan (SL) algorithm and FBP method are based on the convolution of the projection function with filter, followed by integration over all angles (backprojection operator). These operations can be expressed as:

$$g(x, y) = -\frac{1}{2\pi^2}\int_0^{\pi} d\xi \int_{-1}^{1} f(\xi, p)\psi(p - p_0)dp, \tag{3}$$

where unknown function support is assumed to be a unit circle, $\psi(p)$ is a Shepp - Logan filter [3]:

$$\psi(kh_p) = \frac{4}{h_p^2(4k^2 - 1)}, k = 0, \pm 1, \pm 2, ..., . \tag{4}$$

Here variable p is discretized as $p_k = kh_p$.

2. Iterative method of Neumann decomposition with Shepp - Logan filter

Iterative Neumann decomposition with Shepp - Logan filter (NDSL algorithm) is a decomposition of operator R^{-1} to a Neumann series, and it can be written as [4] - [6]:

$$g^0 = R_0^{-1} f,\ g^{i+1} = g^i + \lambda^i R_0^{-1} \Delta f^i, \tag{5}$$

where R_0^{-1} is an approximation of inverse Radon transform operator by FBP with SL filter; $\Delta f^i = f - Rg^i$ is a residual vector of the projections at the i-th iteration; λ^i is a relaxation parameter; g^i is an approximate reconstruction for i-th iteration.

The NDSL algorithm algorithm consists of the next steps:
(1) Zero approximation $g^0(x, y)$ is obtained after application FBP to original projections. (2) For current estimation $g^i(x, y)$ of tomogram, we calculate its Radon transform and than residual vector of projections Δf^i, than apply FBP to this vector. (3) New estimation of the tomogram $g^{i+1}(x, y)$ is obtained from previous estimation (step 2) with weight λ^i in (5). Relaxation parameter λ^i was chosen by the conditions of convergence of iterative process. (4) Verifying criterion of ending iteration process, if it is not satisfied then go to the step 2.

3. Numerical Simulations

In our numerical simulations we have used several model functions (or phantoms) taken from the tomographical Topas-Micro library [7]. First model was a two-dimensional elliptical gaussian defined by:

$$g(x, y) = C \exp[-4 \ln 2\ t^2], \tag{6}$$

$$t^2 = \frac{((x - x_0)\cos\varphi + (y - y_0)\sin\varphi)^2}{a^2} + \frac{(-(x - x_0)\sin\varphi + (y - y_0)\cos\varphi)^2}{b^2},$$

where φ - is an angle of model rotation relatively to OX axis, a and b - are two full widths of this phantom, x_0, y_0 - are coordinates of its center, C - its amplitude. Model of an elliptical parabola of degree λ ($\lambda > 0$) is given by [1]:

$$g(x, y) = \begin{cases} C(1 - t^2)^\lambda, & t < 1\ , \\ 0, & t \geq 1. \end{cases} \tag{7}$$

Parameter t here is the same as in (6).

Fig.1 shows: (a) model No.220, which is a shifted rotated gaussian; b) model No.221 - an axisymmetrical central gaussian; c) model No.216 - a shifted rotated parabola; d) model No.209 - a shifted rectangular.

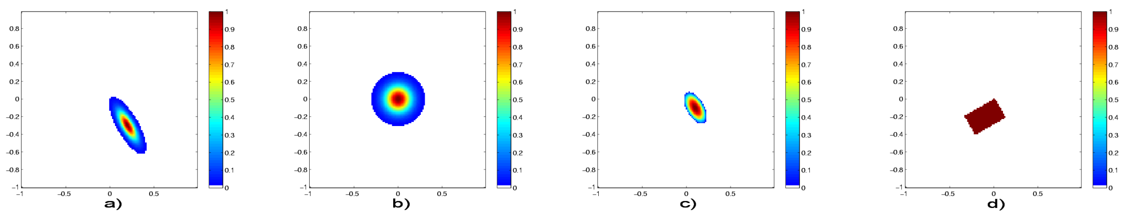


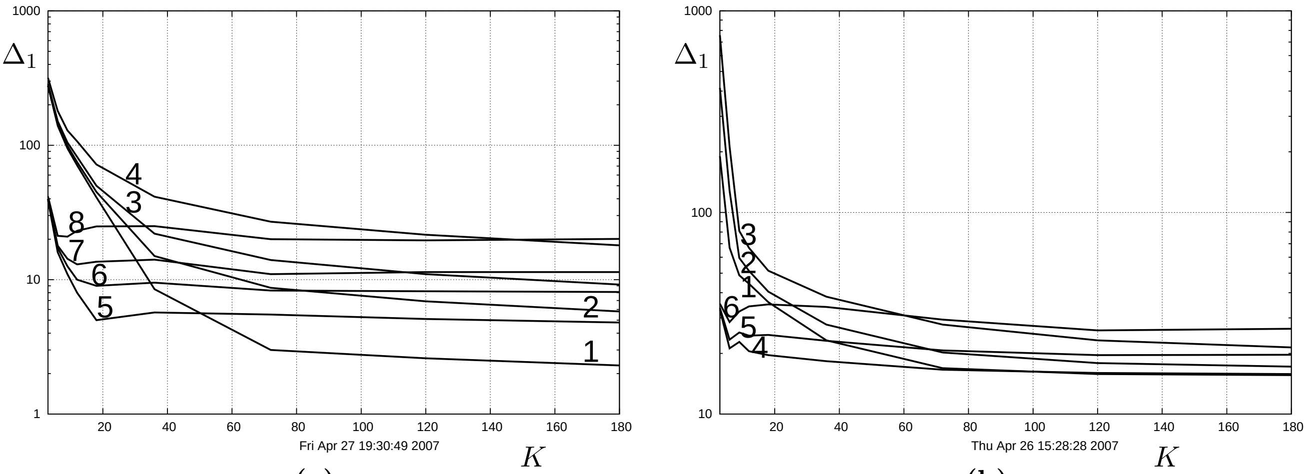
Figure 1: Elementary phantoms.

As a reconstruction mean-square error estimation we have used the next relative norm Δ_1:

$$\Delta_1 = \sqrt{\frac{\sum\sum(g_{ij} - g_{ij}^{exa})^2}{\sum\sum(g_{ij}^{exa})^2}} \cdot 100, \% \tag{8}$$

where g^{exa} is exact tomogram, g - reconstructed one. Reconstruction errors Δ_1 for two phantoms are presented in Fig.2 as a function of the number of projections for two algorithms: an iterative Gerchberg - Papoulis (GP) [7] and classical SL algorithm.

As Fig. 2(a) indicates, iterative algorithm GP is more stable to the random noise in projections, and have much smaller error Δ_1 in the case of a few projections ($K = 3 - 12$). Classical SL algorithm is working properly for more than 40 projections, and it is not suitable for a few - view tomography problems. With the model 209 in Fig.2(b) we have almost the same results, though here SL algorithm is working worse, because we have a stepwise function here.



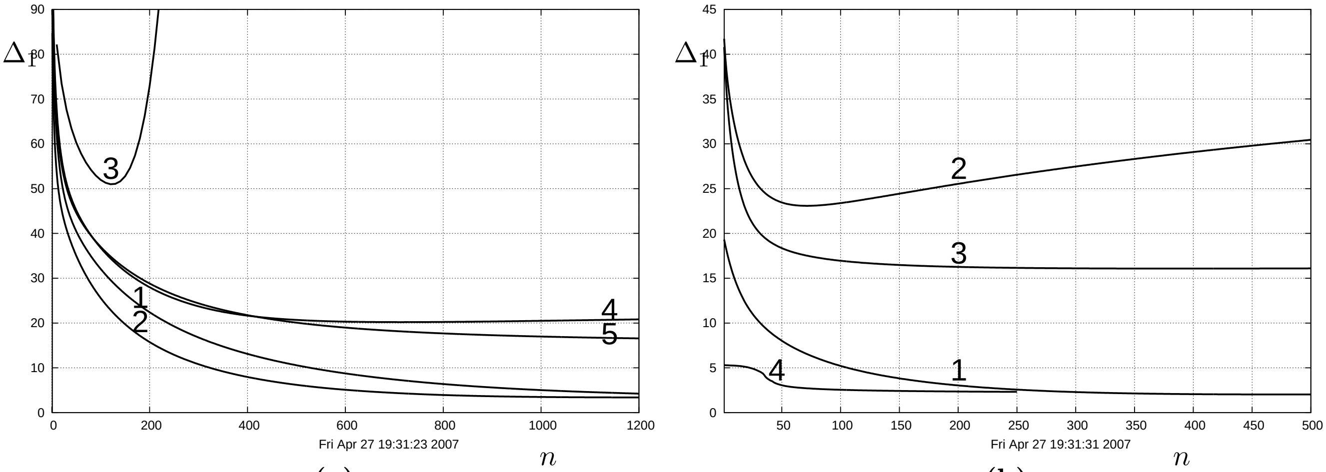
(a) **(b)**

Figure 2: Dependencies of the reconstruction mean-square error Δ_1 on the projections number K for GP and SL algorithms, gaussian model No.220 (a) and rectangular model No.209 (b); a) curves No.1-4: method SL, noise levels $\kappa = 0, 3, 5, 10\%$; curves No.5-8: method GP, $\kappa = 0, 3, 5, 10\%$; b) curves No.1-3: method SL, noise levels $\kappa = 0, 5, 10\%$; curves No.4-6: method GP, $\kappa = 0, 5, 10\%$.

Usually, experimental measurements have random noise in projections. In the next numerical simulations, we made the smoothing of the sinogram by filtering the noise of projections in the Fourier space. It was done by Tikhonov regularization with choosing of regularization parameter α_{res} by residual criterion [8].

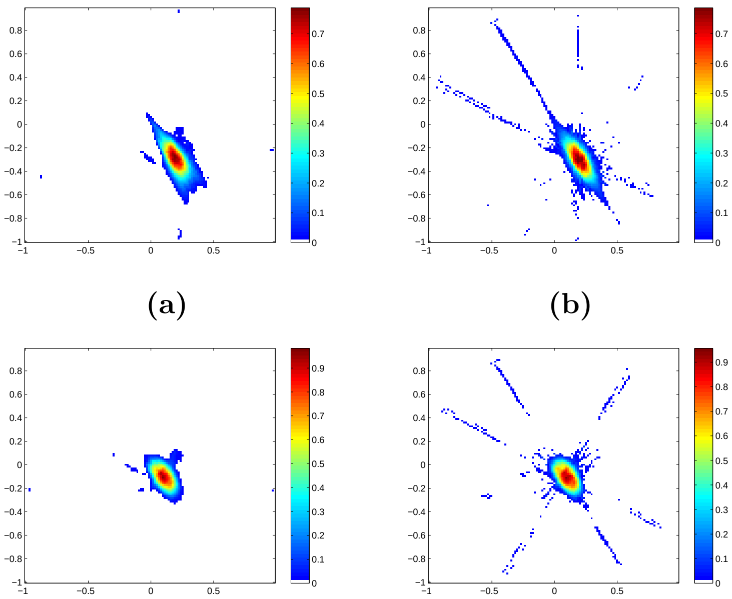
Lets consider the iterative NDSL algorithm (5) with addition of random noise to projections, followed by its smoothing. This algorithm is depicted in Fig.3 (a), (b) and shows fast convergence and definitely less reconstruction error, than classic (non iterative) SL. In Fig.3(a) we can see the error dependencies for different relaxation parameters λ. Wrong choice of λ for the curve 3 demonstrates a fast divergence. We can see, that even for 7 projections a small reconstruction error has taken place, $\Delta_1 = 4\%$ (curve 2, Fig.3(a)). This result is very close to GP algorithm result with $k = 7$, which is one of the best methods in a few - view tomography. After adding 10% noise to projections (curve 4), algorithm is slowly start to diverge, but after smoothing of the projections (curve 5) we see some improvement: the reconstruction error is smaller, divergence is disappeared. In the case of the larger number of projections, for example $K = 19$ in Fig.3(b), we can obtain a better reconstruction, $\Delta_1 = 2\%$ (curve 1). With 10% noise in data the error is larger than in $k = 7$ case. It can be explained by increasing amount of noise in the projections. Thus, implementation of smoothing procedure is necessary. GP iterative algorithm (curve 4 in the Fig. (3(b)) shows very small reconstruction error on the first iteration $\Delta = 5.2\%$, in comparison with iterative SL algorithm that have error $\Delta_1 = 19\%$ at the same iteration. However, iterative SL algorithm have a slow convergence, but finally the reconstruction error is almost equal to GP one. The main conclusion here is that GP algorithm more convenient in use, when we are strongly limited by computer time.

In Fig.4 we can see reconstructions obtained from iterative GP (a, c) and NDSL (b, d) algorithms with 10% noise in smoothed projections. Despite of small artifacts level in the GP reconstructions, Δ_1 here is



(a) **(b)**

Figure 3: Dependencies of the reconstruction mean-square error Δ_1 on the iteration number n for NDSL algorithm, parabola model No.216: a) $K = 7$, curves No.1-3: $\kappa = 0\%, \lambda = 0.03, 0.05, 0.06$; No.4: $\kappa = 10\%, \lambda = 0.03$; No.5: smoothed projections, $\kappa = 10\%, \lambda = 0.03$; b) $K = 19, \lambda = 0.03$, curves No.1-2: $\kappa = 0, 10\%$; No.3: smoothed projections, $\kappa = 10\%$; No.4: iterative GP algorithm, $\kappa = 0\%$.



(c) **(d)**

Figure 4: Reconstruction of the gaussian model No.220 (a, b) and the parabola model No.216 (c, d), $K = 7, \kappa = 10\%$ with smoothed projections: a) GP, 120 iterations, $\Delta_1 = 18.9\%$; b) NDSL, 1200 iterations, $\lambda = 0.03, \Delta_1 = 16.5\%$; c) GP, 120 iterations, $\Delta_1 = 13.2\%$; d) NDSL, 1200 iterations, $\lambda = 0.03, \Delta_1 = 13.1\%$;

larger than in the NDSL reconstructions. From the NDSL reconstruction it is seen that the contour of gaussian (Fig.4 (b)) is more smoothed and clear.

Iterative reduction of the residual vector norm in the procedure (5) can also help to solve some problems of image processing. For example, hiding image in the Radon space as a small disturbance like text shown in Fig.5b, we can extract separately consistent sinogram and added 'background'. Fig.5c shows result of such separation by the NDSL iterative algorithm, where the first step is to calculate a trial sinogram from disturbed tomogram like one in Fig.5a . This technique could be useful for steganography applications.

(a) **(b)** **(c)**

Figure 5: Example of tomogram (a) with unknown image (b) hidden in its sinogram; (c) result of extraction of unknown image.

4. Conclusions

In this study, results of comparison of iterative Gerchberg-Papoulis algorithm and method of decomposition of inverse Radon operator R^{-1} to the Neumann series (NDSL) were shown. The implementation of the iterative NDSL algorithm in a few - view tomography problems is quite possible, and its accuracy and noise resistance are comparable with ones of the Gerchberg-Papoulis algorithm, which is one of the most advanced method in this field. Also, by smoothing of highly noisy projections we are able to prevent algorithm's divergence and obtain comparatively small reconstruction errors.

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