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On the Lepskii principle for choice of the regularization parameter

T. Raus*, **U. Hämarik****

* Institute of Applied Mathematics,
University of Tartu, Liivi 2,
50409 Tartu, Estonia
E-mail: Toomas.Raus@ut.ee

** Institute of Applied Mathematics,
University of Tartu, Liivi 2,
630090 Tartu, Estonia
E-mail: Uno.Hamarik@ut.ee

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We consider an operator equation

$$Au = f, \quad f \in R(A), \tag{1}$$

where $A \in L(H,F)$ is the linear continuous operator between real Hilbert spaces H and F. In general our problem is ill-posed: the range R(A) may be non-closed, the kernel N(A) may be non-trivial. We suppose that instead of exact right-hand side f we have only an approximation $f_\delta \in F$, $\|f_\delta - f\| \le \delta$. To get regularized solution u_r of the equation Au = f we consider the regularization methods in the general form (see [15]), using the approximation

$$u_r = \left(I - A^* A g_r(A^*A)\right)u_0 + g_r(A^*A)A^* f_\delta . \tag{2}$$

Here u_0 is the initial approximation, r is the regularization parameter, I is the identity operator and the generating function $g_r(\lambda)$ satisfies the conditions (3) - (5).

$$\sup_{0\le\lambda\le a} \left|\sqrt{\lambda}\, g_r(\lambda)\right| \le \gamma_* \sqrt{r}, \quad r \ge 0, \tag{3}$$

$$\sup_{0\le\lambda\le a} \lambda^{p} \left|1 - \lambda\, g_r(\lambda)\right| \le \gamma_p r^{-p}, \quad r \ge 0, \quad 0 \le p \le p_0, \tag{4}$$

$$\sup_{0\le\lambda\le a} \left|g_r(\lambda)\right| \le \gamma r, \quad r \ge 0. \tag{5}$$

Here p_0, γ, γ_* and γ_p are positive constants, $a \ge \|A^*A\|$, $\gamma_0 \le 1$ and the greatest value of p_0, for which the inequality (4) holds is called the ***qualification of method***.

The following regularization methods are special cases of general method (2).

P1 The Tikhonov method $u_\alpha = \left(\alpha I + A^*A\right)^{-1} A^* f_\delta$. Here

$u_0 = 0,\ r = \alpha^{-1}, g_r(\lambda) = \left(\lambda + r^{-1}\right)^{-1},\ p_0 = 1, \gamma = 1,\ \gamma_* = 1/2, \quad \gamma_p = p^p (1-p)^{1-p}$.

P2 The iterative variant of the Tikhonov method. Let $m \in N$, $m \ge 1$, $u_0 = u_{0,\alpha} \in H$ - initial approximation and $u_{n,\alpha} = \left(\alpha I + A^*A\right)^{-1}\left(\alpha\, u_{n-1,\alpha} + A^* f_\delta\right)$ $(n = 1,2,..,m)$. Here

$$r = \alpha^{-1}, g_r(\lambda) = \frac{1}{\lambda}\left(1 - \left(\frac{1}{1+r\lambda}\right)^m\right),\ p_0 = m, \gamma = m,\ \gamma_* = \sqrt{m},\ \gamma_p = \left(\frac{p}{m}\right)^p \left(1 - \frac{p}{m}\right)^{m-p}.$$

P3 Explicit iteration scheme (the Landweber's method). Let

$u_n = u_{n-1} - \mu A^*\left(Au_{n-1} - f_\delta\right),\ \mu \in \left(0, 1/\|A^*A\|\right],\ n = 1,2,\ldots$. Here

$$r = n,\ g_r(\lambda) = \frac{1}{\lambda}\left(1 - (1-\mu\lambda)^r\right),\ p_0 = \infty, \gamma = \mu,\ \gamma_* = \sqrt{\mu},\ \gamma_p = \left(\frac{p}{\mu e}\right)^p.$$

P4 Implicit iteration scheme. Let $\alpha > 0$ be a constant and

$u_n = \left(\alpha I + A^*A\right)^{-1}\left(\alpha u_{n-1} + A^* f_\delta\right),\ n = 1,2,\ldots$. Here

$r = n, g_r(\lambda) = \dfrac{1}{\lambda}\left(1 - \left(\dfrac{\alpha}{\alpha+\lambda}\right)^r\right),\ p_0 = \infty, \gamma = \dfrac{1}{\alpha},\ \gamma_* = \dfrac{b_0}{\sqrt{\alpha}},$ where $b_0 = \sup\limits_{0<\lambda<\infty} \lambda^{-1/2}\left(1 - e^{-\lambda}\right) \approx 0.6382$ and $\gamma_p = (\alpha\, p)^p$.

P5 The method of the Cauchy problem: approximation u_r solves the Cauchy problem

$u'(r) + A^*Au(r) = A^* f_\delta, \quad u(0) = u_0$. Here $g_r(\lambda) = \dfrac{1}{\lambda}\left(1 - e^{-r\lambda}\right),\ p_0 = \infty, \gamma = 1,\ \gamma_* = b_0,\ \gamma_p = (p/e)^p$.

The main problem in applying regularization methods is the proper choice of the regularization parameter r in dependence on the noise level δ.
The first prominent a posteriori parameter choice rule is the discrepancy principle, where the regularization parameter is chosen so that the discrepancy has the same order as the error of right-hand side of the equation (1).
Dicrepancy principle. Let b_1, b_2 be the constants such that $b_2 \geq b_1 \geq 1$. If $\|Au_0 - f_\delta\| \leq b_2\delta$, then choose $r(\delta) = 0$. In the contrary case choose the parameter $r = r(\delta) > 0$ for which

$$b_1\delta \leq \|Au_r - f_\delta\| \leq b_2\delta .$$

The drawback of the discrepany principle is that in case $u_* - u_0 \in R((A^*A)^{p/2})$ the order optimal error estimate $\|u_r - u_*\| \leq const\,\delta^{p/(p+1)}$ holds only for $p \leq 2p_0 - 1$ but not for all $p \leq 2p_0$ as by a priori choice r. The following a posteriori parameter rules give order optimal error estimates for all $p \leq 2p_0$.
The modification of the discrepany principle (MD rule) [3,10]. Let b_1, b_2 be the constants such that $b_2 \geq b_1 > 1$. If $\|B_0(Au_0 - f_\delta)\| \leq b_2\delta$, then choose $r(\delta) = 0$. In the contrary case choose the parameter $r = r(\delta) > 0$ for which

$$b_1\delta \leq \|B_r(Au_r - f_\delta)\| \leq b_2\delta .$$

Here the operator B_r has the form

$$B_r = \begin{cases} I, & if \quad p_0 = \infty \\ \left(I - AA^* g_r\left(AA^*\right)\right)^{1/2p_0}, & if \quad p_0 < \infty \end{cases} .$$

For methods with infinite qualification this rule coincides with the discrepancy principle. For methods with finite qualification we apply to the discrepancy the operator B_r. Note, that for Tikhonov method (m=1) and its iterative variant $u_{r,m}$ it holds $\|B_r(Au_{r,m} - f_\delta)\| = (Au_{r,m+1} - f_\delta, Au_{r,m} - f_\delta)^{1/2}$.
Denote $\varphi(r) = \sqrt{r}\|A^*B_r^2(Au_r - f_\delta)\|$ and consider the following rule [11-13]
Rule R1. Let b_1, b_2 be the constants such that $b_2 \geq b_1 > \tilde{\gamma}_{1/2}$, where $\tilde{\gamma}_{1/2} = \gamma_{1/2}$ for methods $p_0 = \infty$ and $\tilde{\gamma}_{1/2} = \left(\gamma_{p_0/(2p_0+2)}\right)^{1+1/p_0}$ for methods $p_0 < \infty$. If $\varphi(1) \leq b_2\delta$, then choose $r(\delta) = 1$. In the contrary case choose the parameter $r = r(\delta) > 1$ for which

$$\varphi(r) \leq b_2\delta \qquad \text{for each } r \geq r(\delta),$$
$$\varphi(r(\delta)) \geq b_1\delta .$$

The MD rule and the Rule R1 are weakly quasioptimal rules (see [14]) also. Namely, if the parameter $r = r(\delta)$ is chosen according to the MD or R1 rule, then it holds the error estimate

$$\|u_{r(\delta)} - u_*\| \leq C(b_1, b_2) \inf_{r \geq 0} \psi(r) + c_0\delta ,$$

where the function $\psi(r) = \|(I - A^*Ag_r(A^*A))(u_0 - u_*)\| + \gamma_*\sqrt{r}\delta$ is an upper bound of the error of the approximate solution $\|u_r - u_*\|$. Note that the Rule R1 is weakly quasioptimal even then if $Qf \in R(A)$, $f \notin R(A)$, where Q is orthoprojector onto $\overline{R(A)}$ and the equation (1) has only the quasisolution.
In last years the Lepskii principle for a posteriori parameter choice is considered in many papers ([1-2, 6-9]). The approximations u_{r_k} are computed for values $r_0 = \delta^{-2}$ and $r_i = r_0 q^i$ with $q < 1$ (i=1,2,…,m+1) and we choose for the regularization parameter $r(\delta) = r_m$ where m is the first index, for which a certain condition is fulfilled. This condition is in [6,8] **(Lepskii principle 1)**

$$\|u_{r_m} - u_{r_{m+1}}\| > c\gamma_*\sqrt{r_m}\delta \tag{6}$$

and in [9] **(Lepskii principle 2)**

$$\exists j \in 1,...,m : \|u_{r_j} - u_{r_{m+1}}\| > c\gamma_*\sqrt{r_j}\delta , \tag{7}$$

where the constant $c = 4$ and the constant γ_* is determined by inequality (3). In the following we show that the Lepskii principle is closely related to the Rule R1.
It is easy to show that for methods P1-P5 the following inequalities hold for functions $g_r(\lambda)$:

$$\gamma(1-q)r\beta_r(\lambda)(1-\lambda g_r(\lambda)) \le g_r(\lambda) - g_{qr}(\lambda) \le \gamma(1-q)r\beta_{qr}(\lambda)(1-\lambda g_{qr}(\lambda)),\ 0<q<1.$$

Using these inequalities we can show that from (6) follows that

$$\varphi(r) \le \left(c'\gamma_*/\left(\gamma(1-q)\sqrt{q}\right)\right)\delta,\ r \ge r_m,$$

$$\varphi(qr_m) \ge \left(c\gamma_*\sqrt{q}/\left(\gamma(1-q)\right)\right)\delta,$$

where $c' := \left\|u_{r_m} - u_{r_{m+1}}\right\| / \left(\gamma_*\sqrt{r_m}\,\delta\right) > c$.

Now, using the results of [12] we can prove the weak quasioptimality of the Lepskii principle 1.

Theorem 1. Let $Qf \in R(A)$. Let $\|f_\delta - f\| \le \delta$ and the parameter $r(\delta) = r_m$ be chosen according to the Lepskii principle 1 with $c > (1-q)\gamma\tilde{\gamma}_{1/2}/\left(\gamma_*\sqrt{q}\right)$. Then for methods P1-P5 it holds the error estimate

$$\left\|u_{r(\delta)} - u_*\right\| \le C(c,q) \inf_{r\ge 0} \psi(r) + c_0\delta,$$

We see that weak quasioptimality of Lepskii principle holds if we take the constant c in (6) substantially less than 4. For example, if q=1/2, then for Tikhonov method $(1-q)\gamma\tilde{\gamma}_{1/2}/\left(\gamma_*\sqrt{q}\right) = \sqrt{2}\,\tilde{\gamma}_{1/2} = \sqrt{2}\ 3\sqrt{3}/16$ and the reasonable choice of c is $2\tilde{\gamma}_{1/2} = 3\sqrt{3}/8$.

On the other hand, to avoid that the lower bound of the constant c goes to zero, if $q \to 1$, then instead of the condition (6) we recommend the condition

$$\left\|u_{r_m} - u_{r_{m+1}}\right\| > c\gamma_*(1-q)\sqrt{r_m}\,\delta/\sqrt{q} = c\gamma_*(r_m - r_{m+1})\delta/\sqrt{r_{m+1}}$$

with $c > \gamma\tilde{\gamma}_{1/2}/\gamma_*$. Note that the Lepskii principle 1 with the sequence of the parameters $r_i = r_{i-1} - 1,\ i = 1,2,\dots,m+1$ coincides in the case of the proper choice of the constant c with the discrete variant of rule R1: we choose the parameter $r(\delta) = r_m$, for which $\varphi(r_i) \le b\delta$, $i < m$ and $\varphi(r_m) \ge b\delta$.

Using inequality (which holds for methods P1-P5)

$$g_s(\lambda) - g_r(\lambda) \le \gamma(s-r)\beta_s(\lambda)(1-\lambda g_r(\lambda)), \qquad 0 \le r \le s$$

we can show that from (7) follows that

$$\left\|B_{qr_m}\left(Au_{qr_m} - f_\delta\right)\right\| \ge \frac{c\gamma_*}{\gamma(\gamma_1)^{1/2}}\delta,$$

$$\varphi(r) \le \left(c'\gamma_*/\left(\gamma(1-q)\sqrt{q}\right)\right)\delta,\ r \ge r_m.$$

Theorem 2. Let $f \in R(A)$, $\|f_\delta - f\| \le \delta$ and the parameter $r(\delta) = r_m$ be chosen according to the Lepskii principle 2 with $c > \gamma(\gamma_1)^{1/2}/\gamma_*$. Then for methods P1-P5 it holds the error estimate

$$\left\|u_{r(\delta)} - u_*\right\| \le C_1(c,q) \inf_{r\ge 0} \psi(r) + c_1\delta.$$

In the following Table 1 the lower bounds of the constant c for quasioptimality of Lepskii principles are presented.

Table 1.

Method	P1	P2	P3	P4	P5
Lepskii 1	$\frac{1-q}{\sqrt{q}}\frac{3\sqrt{3}}{8}$	$\frac{1-q}{\sqrt{q}}\frac{\sqrt{m}(2m+1)^{m+1/2}}{(2m+2)^{m+1}}$	$\frac{1-q}{\sqrt{q}}\frac{1}{\sqrt{2e}}$	$\frac{1-q}{\sqrt{q}}\frac{1}{\sqrt{2b_0}}$	$\frac{1-q}{\sqrt{q}}\frac{1}{\sqrt{2e}\,b_0}$
Lepskii 2	2	$\left(\frac{m-1}{m}\right)^{(m-1)/2}$	$\frac{1}{\sqrt{e}}$	$\frac{1}{b_0}$	$\frac{1}{\sqrt{e}\,b_0}$

One can show, that if in Tikhonov approximation parameter r is choosed by Lepskii principle and we consider error of Tikhonov approximation as the function of constant c, then this error is the increasing function of c, if r is choosed by Lepskii principle 1 with $c > 2(1-q)/q$ or by Lepskii principle 2 with $c > 2(1-q^{m+1-j})/q$.

In case of approximately given error bound $\tilde{\delta}$ the following variant of the Lepskii principle 1 is useful.

Lepskii principle 3. The approximations u_{r_k} are computed for values $r_0 = 1$ and $r_i = r_0 q^{-i}$ with $q < 1$ (i=1,2,…,m) and we choose for the regularization parameter $r(\tilde{\delta}) = r_m$ where m is the first index, for which $\left\| u_{r_m} - u_{r_{m-1}} \right\| \le c\gamma_* \sqrt{r_m}\tilde{\delta}$.

It can be shown that Lepskii principle 3 is stable parameter choice in this sense that $\left\| u_{r(\tilde{\delta})} - u_* \right\| \to 0$ if only the ratio $\| f_\delta - f \| / \tilde{\delta}$ of the actual error of the right hand-side $\| f_\delta - f \|$ and the supposed error level $\tilde{\delta}$ is bounded in the process $\tilde{\delta} \to 0$.

Note that if the sequence of Tikhonov approximations u_{r_i} is computed, for approximate solution one can take instead of as some u_{r_i} linear combination of n Tikhonov approximations. In case of proper coefficients the resulting approximation has qualification $p_0 = n$ (see [5]).

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