

# Applications of the Tikhonov regularization using reproducing kernels to inverse problems

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**Abstract.** This paper is a survey article based on our recent papers on inverse problems using the Tikhonov regularization and reproducing kernels.

The Tikhonov regularization is a basically important idea and method in numerical analysis, however, the extremal functions in the Tikhonov functionals are represented by using the associated singular values and singular functions for the case of compact operators. So, their representations are restrictive and abstract in a sense. We shall give new representations for the case of general bounded linear operators by using the theory of reproducing kernels and we shall introduce various concrete representations for typical cases. Our representations are both analytical and numerical. We introduce a general theory of the Tikhonov regularization using the theory of reproducing kernels containing error estimates and convergence rates. We shall present the general theory and its concrete results for the typical inverse problem for heat conduction with computer graphs (by the references) as the evidence of the power of our inverse formulas.

## 1. Reproducing kernels

We firstly recall a basic relation between linear mappings in the framework of Hilbert spaces and reproducing kernels. In particular, we can see here why we meet ill-posed problems, indeed, we can see the idea and method for the avoidance of the ill-posed problems in the framework of Hilbert spaces. However, this will be a mathematical theory and as numerical methods, we will need the idea of Tikhonov regularization stated in Section 5. See [5] as a general elementary theory of inverse problems and ill-posed problems.

We consider any positive matrix  $K(p, q)$  on a fixed set  $E$ ; that is, for an abstract set  $E$  and for a complex-valued function  $K(p, q)$  on  $E \times E$ , it satisfies that for any finite points  $\{p_j\}$  of  $E$  and for any complex numbers  $\{C_j\}$ ,

$$\sum_j \sum_{j'} C_j \overline{C_{j'}} K(p_{j'}, p_j) \geq 0.$$

Then, by the fundamental theorem by Moore–Aronszajn, we have:

**Proposition 1.1** ([1]) *For any positive matrix  $K(p, q)$  on  $E$ , there exists a uniquely determined functional Hilbert space  $H_K(\text{RKHS}H_K)$  comprising functions  $\{f\}$  on  $E$  and admitting the reproducing kernel  $K(p, q)$  satisfying and characterized by*

$$K(\cdot, q) \in H_K \text{ for any } q \in E \tag{1}$$

and, for any  $q \in E$  and for any  $f \in H_K$

$$f(q) = (f(\cdot), K(\cdot, q))_{H_K}. \quad (2)$$

For some general properties for reproducing kernel Hilbert spaces and for various constructions of the RKHS  $H_K$  from a positive matrix  $K(p, q)$ , see the recent book [23] and its Chapter 2, Section 5, respectively.

## 2. Connections with linear mappings

Let us connect linear mappings in the framework of Hilbert spaces with reproducing kernels ([19]).

For an abstract set  $E$  and for any Hilbert (possibly finite-dimensional) space  $\mathcal{H}$ , we shall consider an  $\mathcal{H}$ -valued function  $\mathbf{h}$  on  $E$

$$\mathbf{h}: E \longrightarrow \mathcal{H} \quad (3)$$

and the linear mapping for  $\mathcal{H}$

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}} \quad \text{for } \mathbf{f} \in \mathcal{H} \quad (4)$$

into a linear space comprising functions on  $E$ . For this linear mapping (4), we form the positive matrix  $K(p, q)$  on  $E$  defined by

$$K(p, q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathcal{H}} \quad \text{on } E \times E. \quad (5)$$

Then, we have the following fundamental results:

(I) For the RKHS  $H_K$  admitting the reproducing kernel  $K(p, q)$  defined by (5), the images  $\{f(p)\}$  by (4) for  $\mathcal{H}$  are characterized as the members of the RKHS  $H_K$ .

(II) In general, we have the inequality in (4)

$$\|f\|_{H_K} \leq \|\mathbf{f}\|_{\mathcal{H}}, \quad (6)$$

however, for any  $f \in H_K$  there exists a uniquely determined  $\mathbf{f}^* \in \mathcal{H}$  satisfying

$$f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathcal{H}} \quad \text{on } E \quad (7)$$

and

$$\|f\|_{H_K} = \|\mathbf{f}^*\|_{\mathcal{H}}. \quad (8)$$

In (6), the isometry holds if and only if  $\{\mathbf{h}(p); p \in E\}$  is complete in  $\mathcal{H}$ .

(III) We can obtain the inversion formula for (7) in the form

$$f \longrightarrow \mathbf{f}^*, \quad (9)$$

by using the RKHS  $H_K$ .

However, this inversion formula will depend on, case by case, the realizations of the RKHS  $H_K$ .

(IV) Conversely, if we have an isometric mapping  $\tilde{L}$  from a RKHS  $H_K$  admitting a reproducing kernel  $K(p, q)$  on  $E$  onto a Hilbert space  $\mathcal{H}$ , then the mapping is linear and its isometric inversion  $\tilde{L}^{-1}$  is represented in the form (4). Here, the Hilbert space  $\mathcal{H}$ -valued function  $\mathbf{h}$  satisfying (3) and (4) is given by

$$\mathbf{h}(p) = \tilde{L}K(\cdot, p) \text{ on } E \quad (10)$$

and, then  $\{\mathbf{h}(p); p \in E\}$  is complete in  $\mathcal{H}$ .

When (4) is isometrical, sometimes we can use the isometric mapping for a realization of the RKHS  $H_K$ , conversely — that is, if the inverse  $L^{-1}$  of the linear mapping (4) is known, then we have  $\|f\|_{H_K} = \|L^{-1}f\|_{\mathcal{H}}$ .

We shall state some general applications of the results (I)~(IV) to several wide subjects and their basic references:

(1) Linear mappings ([21],[23]).

The fact that the image spaces of linear mappings in the framework of Hilbert spaces are characterized as reproducing kernel Hilbert spaces defined by (5) is the most important one in the general theory of reproducing kernels. Therefore, the fact will mean that the theory of reproducing kernels is fundamental and a general concept in mathematics. To look for the characterization of the image space is a starting point when we consider the linear equation (4). (II) gives a generalization of the Pythagorean theorem (see also [18]) and means that in the general linear mapping (4) there exists essentially an isometric identity between the input and the output. (III) gives a generalized (natural) inverse (solution) of the linear mapping (equation) (4). (IV) gives a general method determining and constructing the linear system from an isometric relation between outputs and inputs by using the reproducing kernel in the output space.

(2) Linear mappings among smooth functions ([28]).

We considered linear mappings in the framework of Hilbert spaces, however, we can consider linear mappings in the framework of Hilbert spaces comprising smooth functions, similarly. Conversely, reproducing kernel Hilbert spaces are considered as the images of some Hilbert spaces by considering some decomposed representations (5) of the reproducing kernels. Such decomposition is, in general, possible. This idea is important in [29] and also in the following items (6) and (7).

(3) Nonharmonic linear mappings ([21]).

If the linear system vectors  $\mathbf{h}(p)$  move in a small way (perturbation of the linear systems) in the Hilbert space  $\mathcal{H}$ , then we can not calculate the related positive matrix (5), however, we can discuss the inversion formula and an isometric identity of the linear mapping. The prototype result is the Paley-Wiener theorem on nonharmonic Fourier series.

(4) Various norm inequalities ([24],[26]).

Relations among positive matrices correspond to those of the associated reproducing kernel Hilbert spaces, by the minimum principle. So, we can derive various norm inequalities among reproducing kernel Hilbert spaces. We were able to derive many beautiful norm inequalities.

(5) Nonlinear mappings ([26],[24]).

In a very general nonlinear mapping of a reproducing kernel Hilbert space, we can look for a natural reproducing kernel Hilbert space containing the image space and furthermore, we can derive a natural norm inequality in the nonlinear mapping. How to catch nonlinearity in connection with linearity? It seems that the theory of reproducing kernels gives a fundamental and interesting answer for this question.

(6) Linear (singular) integral equations ([29],[9]).

- (7) Linear differential equations with variable coefficients ([29],[32]).  
In linear integro-differential equations with general variable coefficients, we can discuss the existence and construction of the solutions, if the solutions exist. This method is called a backward transformation method and by reducing the equations to Fredholm integral equations of the first kind and we can discuss the classical solutions, in very general linear equations.
- (8) Approximation theory ([3],[23]).  
Reproducing kernel Hilbert spaces are very nice function spaces, because the point evaluations are continuous. Then, the reproducing kernels are a fundamental tool in the related approximation theory.
- (9) Representations of inverse functions ([25],[23]).  
For any mapping, we discussed the problem of representing its inverse in terms of the direct mapping and we derived a unified method for this problem. As a simple example, we can represent the Taylor coefficients of the inverse of the Riemann mapping function on the unit disc on the complex plane in terms of the Riemann mapping function. This fact was important in the representation of analytic functions in terms of local data in ([30]).
- (10) Various operators among Hilbert spaces ([27]).  
Among various abstract Hilbert spaces, we can introduce various operators of sum, product, integral and derivative by using the linear mapping or very general nonlinear mappings. The prototype operator is convolution and we discussed it from a wide and general viewpoint with concrete examples.
- (11) Sampling theorems ([23], Chapter 4, Section 2).  
The Whittaker-Kotel'nikov-Shannon sampling theorem may be interpreted by (I) and (II), very well and we can discuss the truncation errors in the sampling theory. J. R. Higgins [7] established a fully general theory for [23].
- (12) Interpolation problems of Pick-Nevanlinna type ([22],[21]).  
General and abstract theory of interpolation problems of Pick-Nevanlinna type may be discussed by using the general theory of reproducing kernels.
- (13) Analytic extension formulas and their applications ([6],[30]).  
We were able to obtain various analytic extension formulas and their applications from various isometric identities (II). For their applications to nonlinear partial differential equations, see the survey article by N. Hayashi [6].
- (14) Inversions of a family of bounded linear operators on a Hilbert space into various Hilbert spaces ([31]).

### 3. Best approximation problems

We shall consider the best approximate problem

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} \quad (11)$$

for a vector  $\mathbf{d}$  in  $\mathcal{H}$ . Then, we have

**Proposition 3.1** ([3],[23]) *For a vector  $\mathbf{d}$  in  $\mathcal{H}$ , there exists a function  $\tilde{f}$  in  $H_K$  such that*

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} = \|L\tilde{f} - \mathbf{d}\|_{\mathcal{H}} \quad (12)$$

*if and only if, for the RKHS  $H_K$  admitting the reproducing kernel defined by*

$$k(p, q) = (L^*LK(\cdot, q), L^*LK(\cdot, p))_{H_K}, \quad (13)$$

$$L^* \mathbf{d} \in H_k. \quad (14)$$

Furthermore, if the best approximation  $\tilde{f}$  satisfying (12) exists, then there exists a unique extremal function  $f_{\mathbf{d}}$  with the minimum norm in  $H_K$ , and the function  $f_{\mathbf{d}}$  is expressible in the form

$$f_{\mathbf{d}}(p) = (L^* \mathbf{d}, L^* LK(\cdot, p))_{H_k} \quad \text{on } E. \quad (15)$$

In Proposition 3.1, note that

$$(L^* \mathbf{d})(p) = (L^* \mathbf{d}, K(\cdot, p))_{H_K} = (\mathbf{d}, LK(\cdot, p))_{\mathcal{H}}; \quad (16)$$

that is,  $L^* \mathbf{d}$  is expressible in terms of the known  $\mathbf{d}, L, K(p, q)$  and  $\mathcal{H}$ .  $f_{\mathbf{d}}$  in (15) is the Moore-Penrose generalized inverse solution  $L^\dagger \mathbf{d}$  of the equation  $Lf = \mathbf{d}$ . Therefore, Proposition 3.1 gives a necessary and sufficient condition for the existence of the Moore-Penrose generalized inverse. Proposition 3.1 is rigid and is not practical in practical applications, because, practical data contain noises or errors and the criteria (14) is not suitable. So, we shall consider the Tikhonov regularization and we shall establish a good relation between the Tikhonov regularization and the theory of reproducing kernels.

#### 4. Spectral theory

In order to discuss operator equations for general bounded linear operators  $L$ , following [4] we shall fix the well-established theory among spectral theory, the Moore-Penrose generalized inverse and the Tikhonov regularization. See [5] for the corresponding results for compact operators  $L$ .

Let  $\{E_\lambda\}$  be a spectral family for the self-adjoint operator  $L^*L$ . If  $L^*L$  is continuously invertible, then

$$(L^*L)^{-1} = \int \frac{1}{\lambda} dE_\lambda.$$

In this case, the Moore-Penrose generalized inverse (15) can be represented by the Gaussian normal equation

$$f_{\mathbf{d}}(p) = \int \frac{1}{\lambda} dE_\lambda L^* \mathbf{d}. \quad (17)$$

If  $\mathcal{R}(L)$  is non-closed and  $\mathbf{d} \notin \mathcal{D}(L^\dagger)$ , i.e. if the equation  $Lf = \mathbf{d}$  is ill-posed, then the integral in (17) does not exist. Then, we shall define

$$f_{\mathbf{d}, \alpha}(p) = \int \frac{1}{\lambda + \alpha} dE_\lambda L^* \mathbf{d}. \quad (18)$$

By construction, the operator on the right-hand side of (18) acting on  $\mathbf{d}$  is continuous, so that, for noisy data  $\mathbf{d}^\delta$  with  $\|\mathbf{d} - \mathbf{d}^\delta\|_{\mathcal{H}} \leq \delta$ , we can bound the error between  $f_{\mathbf{d}, \alpha}$  and

$$f_{\mathbf{d}, \alpha}^\delta(p) = \int \frac{1}{\lambda + \alpha} dE_\lambda L^* \mathbf{d}^\delta \quad (19)$$

as follows:

**Proposition 4.1** ([4], pages 71-73) For any  $\mathbf{d} \in \mathcal{D}(L^\dagger)$ ,

$$\lim_{\alpha \rightarrow 0} \frac{1}{L^*L + \alpha I} L^* \mathbf{d} = \lim_{\alpha \rightarrow 0} f_{\mathbf{d}, \alpha} = f_{\mathbf{d}}. \quad (20)$$

Furthermore,

$$\|Lf_{\mathbf{d}, \alpha} - Lf_{\mathbf{d}, \alpha}^\delta\|_{\mathcal{H}} \leq \delta \quad (21)$$

and

$$\|f_{\mathbf{d},\alpha} - f_{\mathbf{d},\alpha}^\delta\|_{H_K} \leq \frac{\delta}{\sqrt{\alpha}}. \quad (22)$$

**Proposition 4.2** ([4], pages 117-118) *For any  $\mathbf{d} \in \mathcal{D}(L^\dagger)$  with  $\|\mathbf{d} - \mathbf{d}^\delta\|_{\mathcal{H}} \leq \delta$ , the function  $f_{\mathbf{d},\alpha}^\delta$  defined by (19) is the unique minimizer of the Tikhonov functional*

$$\inf_{f \in H_K} \{\alpha \|f\|_{H_K}^2 + \|\mathbf{d}^\delta - Lf\|_{\mathcal{H}}^2\}. \quad (23)$$

If  $\alpha = \alpha(\delta)$  is such that

$$\lim_{\delta \rightarrow 0} \alpha(\delta) = 0$$

and

$$\lim_{\delta \rightarrow 0} \frac{\delta^2}{\alpha(\delta)} = 0,$$

then

$$\lim_{\delta \rightarrow 0} f_{\mathbf{d},\alpha}^\delta = f_{\mathbf{d}} = L^\dagger(\mathbf{d}). \quad (24)$$

## 5. Representation of the extremal functions in Tikhonov regularization

Our main purpose here is to give an effective representation of the extremal functions  $f_{\mathbf{d},\alpha}$  or  $f_{\mathbf{d},\alpha}^\delta$  in the Tikhonov regularization, since the representation by spectral theory is abstract, in many practical problems.

We set

$$K_L(\cdot, p; \alpha) = \frac{1}{L^*L + \alpha I} K(\cdot, p).$$

Then, by introducing the inner product, for any fixed positive  $\alpha > 0$

$$(f, g)_{H_K(L; \alpha)} = \alpha(f, g)_{H_K} + (Lf, Lg)_{\mathcal{H}}, \quad (25)$$

we shall construct the Hilbert space  $H_K(L; \alpha)$  comprising functions of  $H_K$ . This space, of course, admits a reproducing kernel. Furthermore, we obtain, directly

**Proposition 5.1** ([33]) *The extremal function  $f_{\mathbf{d},\alpha}(p)$  in the Tikhonov regularization*

$$\inf_{f \in H_K} \{\alpha \|f\|_{H_K}^2 + \|\mathbf{d} - Lf\|_{\mathcal{H}}^2\} \quad (26)$$

*is represented in terms of the kernel  $K_L(p, q; \alpha)$  as follows:*

$$f_{\mathbf{d},\alpha}(p) = (\mathbf{d}, LK_L(\cdot, p; \alpha))_{\mathcal{H}} \quad (27)$$

*where the kernel  $K_L(p, q; \alpha)$  is the reproducing kernel for the Hilbert space  $H_K(L; \alpha)$  and it is determined as the unique solution  $\tilde{K}(p, q; \alpha)$  of the equation:*

$$\tilde{K}(p, q; \alpha) + \frac{1}{\alpha} (L\tilde{K}_q, LK_p)_{\mathcal{H}} = \frac{1}{\alpha} K(p, q) \quad (28)$$

with

$$\tilde{K}_q = \tilde{K}(\cdot, q; \alpha) \in H_K \quad \text{for } q \in E, \quad (29)$$

and

$$K_p = K(\cdot, p) \in H_K \quad \text{for } p \in E.$$

In (27), when  $\mathbf{d}$  contains errors or noises, we need its error estimate. For this, we can obtain the general result:

**Proposition 5.2** ([17],[9]). *In (17), we obtain the estimate*

$$|f_{\mathbf{d},\alpha}(p)| \leq \frac{1}{\sqrt{\alpha}} \sqrt{K(p,p)} \|\mathbf{d}\|_{\mathcal{H}}.$$

For many concrete applications of these general theorems, see, for example, [2,9,11-17,32-35].

## 6. A typical example for the inversion of the heat conduction

We shall give surprisingly simple approximate real inversion formulas for the Gaussian convolution (the Weierstrass transform)

$$u_F(x, t) = (L_t F)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} F(\xi) \exp \left\{ -\frac{|\xi - x|^2}{4t} \right\} d\xi \quad (30)$$

for the functions of  $L_2(\mathbf{R}^n)$ . This integral transform which represents the solution  $u(x, t)$  of the heat equation

$$u_t(x, t) = u_{xx}(x, t) \quad \text{on} \quad \mathbf{R}^n \times \{t > 0\}$$

satisfying the initial condition

$$u(x, 0) = F(x) \quad \text{on} \quad \mathbf{R}^n,$$

is very fundamental and has many applications to mathematical sciences. See the recent article [39] and its references for its many significants and applications to medical science and physics.

Over twenty years ago, in the one dimensional case  $n = 1$ , the author [20] gave a surprise characterization of the image  $u_F(x, t)$  of (30) for  $L_2(\mathbf{R}) = L_2(\mathbf{R}, dx)$  functions in terms of an analytic function and established a very simple complex inversion formula following the idea (I). The paper created a new method and many applications to general integral transforms in the framework of Hilbert spaces and analytic extension formulas. See, for example [2] and [23], and their many references. However, in particular, its real inversion formulas are very involved, for example, recall that:

For a bounded and continuous function  $F(x)$  and for  $t = 1$ ,

$$e^{-D^2}[(L_1 F)(x)] = F(x) \quad \text{pointwisely} \quad \text{on} \quad \mathbf{R}$$

([8], p. 182). So, one might think that its real inversion formulas will be essentially involved for catching "analyticity" in terms of the data on the real line as in the real inversion formulas of the Laplace transform. See also [10] for a recent related article.

Indeed, this inverse problem is very famous as a typical ill-posed problem that is very difficult.

In those papers [33,15], however we were able to obtain simple and practical approximate real inversion formulas by the method in Section 5 using the Sobolev reproducing Hilbert spaces. Furthermore, we illustrated their numerical experiments by using computers and we can realize that we were able to obtain practical real inversion formulas.

In [17], we applied the Paley-Wiener spaces as the reproducing kernel Hilbert spaces in the above theory and we got a surprisingly improved version in the sense of numerical.

At first we shall fix notations and basic results in the Paley-Wiener spaces following the book by F. Stenger[38] and at the same time we shall show the basic relation of the sampling theory and the theory of reproducing kernels.

We shall consider the integral transform, for  $L_2(\mathbf{R}^n, (-\pi/h, +\pi/h)^n)$ ,  $(h > 0)$  functions  $g$

$$f(z) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \chi_h(t) g(t) e^{-iz \cdot t} dt. \quad (31)$$

Here,  $z = (z_1, z_2, \dots, z_n)$ ,  $t = (t_1, t_2, \dots, t_n)$ ,  $dt = dt_1 \cdot dt_2 \cdots dt_n$ ,  $z \cdot t = z_1 t_1 + \cdots + z_n t_n$  and

$$\chi_h(t) = \prod_{\nu=1}^n \chi(t_\nu, (-\pi/h, +\pi/h)),$$

the characteristic function  $\chi$  of  $(-\pi/h, +\pi/h)$ . In order to identify the image space following the idea (I), we form the reproducing kernel

$$\begin{aligned} K_h(z, \bar{u}) &= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \chi_h(t) e^{-iz \cdot t} \overline{e^{-iu \cdot t}} dt \\ &= \prod_{\nu=1}^n \frac{1}{\pi(z_\nu - \bar{u}_\nu)} \sin \frac{\pi}{h} (z_\nu - \bar{u}_\nu). \end{aligned} \quad (32)$$

The image space of (31) is called the Paley-Wiener space  $W(\frac{\pi}{h})$  ( $:= W_h$ ) comprised of all analytic functions of exponential type satisfying, for each  $\nu$ , for some constant  $C_\nu$  and as  $z_\nu \rightarrow \infty$

$$|f(z_1, \dots, z_\nu, z_{\nu+1}, \dots, z_n)| \leq C_\nu \exp\left(\frac{\pi|z_\nu|}{h}\right)$$

and

$$\int_{\mathbf{R}^n} |f(x)|^2 dx < \infty.$$

From the identity, for multi-index  $j = (j_1, j_2, \dots, j_n)$

$$K_h(jh, j'h) = \prod_{\nu=1}^n \frac{1}{h} \delta(j_\nu, j'_\nu)$$

(the Kronecker's  $\delta$ ), for each  $\nu$ , since  $\delta(j_\nu, j'_\nu)$  is the reproducing kernel for the Hilbert space  $\ell^2$ , from (II) and the Parseval's identity we have the isometric identities in (31)

$$\begin{aligned} \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} |g(t)|^2 dt &= h^n \sum_j |f(jh)|^2 \\ &= \int_{\mathbf{R}^n} |f(x)|^2 dx. \end{aligned}$$

That is, the reproducing kernel Hilbert space  $H_{K_h}$  with  $K_h(z, \bar{u})$  is characterized as a space comprising the Paley-Wiener space  $W_h$  and with the norms above in the both senses of discrete and continuous versions. Here we used the well-known result that  $\{jh\}_j$  is a uniqueness set for the Paley-Wiener space  $W_h$ ; that is,  $f(jh) = 0$  for all  $j$  implies  $f \equiv 0$ . Then, the reproducing property of  $K_h(z, \bar{u})$  states that

$$\begin{aligned} f(x) &= (f(\cdot), K_h(\cdot, x))_{H_{K_h}} = h^n \sum_j f(jh) K_h(jh, x) \\ &= \int_{\mathbf{R}^n} f(\xi) K_h(\xi, x) d\xi, \end{aligned}$$

in particular, on the real space  $x$ . This representation is the sampling theorem which represents the whole data  $f(x)$  in terms of the discrete data  $\{f(jh)\}_j$ . Further a general theory for the sampling theory and error estimates for some finite points  $\{hj\}_j$ , see [23].

Following our general theory, we can obtain the concrete results:



**Proposition 6.1** ([17]) *For any function  $g \in L_2(\mathbf{R}^n)$  and for any  $\lambda > 0$ , the best approximate function  $F_{t,\lambda,h,g}^*$  in the sense*

$$\begin{aligned} & \inf_{F \in H_{K_h}} \left\{ \lambda \|F\|_{H_{K_h}}^2 + \|g - u_F(\cdot, t)\|_{L_2(\mathbf{R}^n)}^2 \right\} \\ &= \lambda \|F_{t,\lambda,h,g}^*\|_{H_{K_h}}^2 + \|g - u_{F_{t,\lambda,h,g}^*}(\cdot, t)\|_{L_2(\mathbf{R}^n)}^2 \end{aligned} \quad (33)$$

*exists uniquely and  $F_{t,\lambda,h,g}^*$  is represented by*

$$F_{t,\lambda,h,g}^*(x) = \int_{\mathbf{R}^n} g(\xi) Q_{t,\lambda,h}(\xi - x) d\xi \quad (34)$$

*for*

$$Q_{t,\lambda,h}(\xi - x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{\chi_h(p) e^{-ip \cdot (\xi - x)} dp}{\lambda e^{|p|^2 t} + e^{-|p|^2 t}}.$$

*If, for  $F \in H_{K_h}$  we consider the output  $u_F(x, t)$  and we take  $u_F(\xi, t)$  as  $g$ , then we have the result: as  $\lambda \rightarrow 0$*

$$F_{t,\lambda,h,g}^* \rightarrow F, \quad (35)$$

*uniformly.*

Here we note the surprising fact that for the Sobolev space case, for  $\lambda = 0$  the corresponding representation (34) does not exist ([33],[15]), meanwhile for the Paley-Wiener space  $W(\frac{\pi}{h})$  case of (34), for  $\lambda = 0$  the representation (34) is still valid; that is, in Proposition 6.1, the result is valid for even  $\lambda = 0$ . Hence, we can consider the results for  $\lambda = 0$  in the spirit of the Tikhonov regularization in which we are interested in a small  $\lambda$  or  $\lambda$  tending to zero. That is, when we use the Paley-Wiener space  $W(\frac{\pi}{h})$ , we need not to consider the Tikhonov regularization. Then, since

$$(L_t F_{t,0,h,g}^*)(x) = (g(\cdot), K_h(\cdot, x))_{L_2(\mathbf{R}^n)}$$

as we see from (32) and since the output is the orthogonal projection of  $g$  onto the Paley-Wiener space  $W(\frac{\pi}{h})$ , we can estimate the difference of the output of our inverse  $F_{t,0,h,g}^*$  and  $g$ , clearly as

$$\|L_t F_{t,0,h,g}^* - g\|_{L_2(\mathbf{R}^n)}$$

which is the distance from  $g$  onto the Paley-Wiener space  $W(\frac{\pi}{h})$ . Of course,  $F_{t,0,h,g}^*$  is the Moore-Penrose generalized inverse of the operator equation, for any  $g \in L_2(\mathbf{R})$  and  $F \in W(\frac{\pi}{h})$ ,

$$L_t F = g.$$

For the Paley-Wiener space  $W(\frac{\pi}{h})$ , we need not to use the Tikhonov regularization and we can look for the Moore-Penrose generalized inverse  $F_{t,0,h,g}^*$  by using the theory of reproducing kernels ([23], pp. 178-180). However, we had better to calculate the extremal functions  $F_{t,\lambda,h,g}^*$  in the Tikhonov regularization and to set  $\lambda = 0$ , because the structure of the Moore-Penrose generalized inverses is involved.

Meanwhile, we are interested in the convergence property of the extremal functions (34) when  $h$  tends to zero for a general  $g$ .

However, we can see directly that in (34)

$$\lim_{h \rightarrow 0} (L_t F_{t,0,h,g}^*)(x) = g(x)$$

at the points  $x$  where  $g(x)$  is continuous.

We are interested in numerical experiments for the both cases of the Sobolev space  $H_S$  and the Paley-Wiener space  $W(\frac{\pi}{h})$  for their convergences as  $\lambda \rightarrow 0$  and  $h \rightarrow 0$ . For the Sobolev spaces and  $\lambda \rightarrow 0$ , see [15].

The real inversion formula (34) will give a practical formula for the Gaussian convolution. We will show experimental results by using computers in Figures. In [15], we stated that when we use the Sobolev spaces as the reproducing kernels, *we will see that in order to overcome the high "ill-posedness" in the real inversion and in order to catch "analyticity" of the image of (30) we must work hardly; that is, we must take a very small  $\lambda$  and we must calculate the corresponding integral (34) hardly in the sense of numerical. Computers help us this hard work to calculate the integral for a very small  $\lambda$ .* However, when we use the Paley-Wiener spaces for  $\lambda = 0$  and for  $h$  tending to 0, we can obtain a very surprising simple and improved inversion formula.

For any  $\lambda > 0$  and any  $t > 0$ , we shall define a linear mapping

$$M_{t,\lambda,h} : L_2(\mathbf{R}^n) \rightarrow H_{K_h}$$

by  $M_{t,\lambda,h}(g) = F_{t,\lambda,h,g}^*$ . Now, we consider the composite operators  $L_t M_{t,\lambda,h}$  and  $M_{t,\lambda,h} L_t$ . Using Fourier's integrals it can be shown that for  $F \in H_{K_h}$ ,

$$\begin{aligned} (M_{t,\lambda,h} L_t F)(x) &= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \left\{ F(\xi) \right. \\ &\quad \cdot \left. \int_{\mathbf{R}^n} \frac{\chi_h(p) e^{-ip \cdot (\xi - x)} dp}{\lambda e^{2|p|^2 t} + 1} \right\} d\xi \end{aligned} \quad (36)$$

and for  $g \in L_2(\mathbf{R}^n)$ ,

$$\begin{aligned} (L_t M_{t,\lambda,h} g)(x) &= \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \left\{ g(\xi) \right. \\ &\quad \cdot \left. \int_{\mathbf{R}^n} \frac{\chi_h(p) e^{-ip \cdot (\xi - x)} dp}{\lambda e^{2|p|^2 t} + 1} \right\} d\xi. \end{aligned} \quad (37)$$

Setting

$$\Delta_{t,\lambda,h}(x - \xi) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{\chi_h(p) e^{-ip \cdot (\xi - x)} dp}{\lambda e^{2|p|^2 t} + 1}$$

in (36) and (37), we have

$$(M_{t,\lambda,h} L_t F)(x) = \int_{\mathbf{R}^n} F(\xi) \Delta_{t,\lambda,h}(x - \xi) d\xi, \quad (F \in H_{K_h}) \quad (38)$$

and

$$(L_t M_{t,\lambda,h} g)(x) = \int_{\mathbf{R}^n} g(\xi) \Delta_{t,\lambda,h}(x - \xi) d\xi, \quad (g \in L_2(\mathbf{R}^n)). \quad (39)$$

Then we obtain that

$$\lim_{h \rightarrow 0} \Delta_{t,0,h}(x - \xi) = \delta(x - \xi), \quad (40)$$

$$\lim_{h \rightarrow 0} M_{t,0,h} L_t = I \quad (41)$$

and

$$\lim_{h \rightarrow 0} L_t M_{t,0,h} = I. \quad (42)$$

The precise meanings of (41) and (42) are given as follows:

$$\lim_{h \rightarrow 0} (M_{t,0,h} L_t F)(x) = F(x)$$

$$\lim_{h \rightarrow 0} L_t M_{t,0,h} g = g \quad (43)$$

at the points where  $F$  and  $g$  are continuous.

In the Figures 1-3 in [17],  $n = 2$  and  $g$  is the characteristic function  $\chi_h(\xi)$ . The above Figures in 1-3 are the outputs of (34) for  $\lambda = 0, t = 1$  for various  $h$  and the lower Figures are their outputs by  $L_t$ .

We used the direct representation of (34)

$$\begin{aligned} F_{t,0,h,g}^*(x) &= \int_{-\pi/h}^{\pi/h} \frac{e^{p_1^2 t} \cos(p_1 x_1) \sin p_1}{\pi p_1} dp_1 \\ &\cdot \int_{-\pi/h}^{\pi/h} \frac{e^{p_2^2 t} \cos(p_2 x_2) \sin p_2}{\pi p_2} dp_2 \\ &= F_{t,0,h,g}^{*(1)}(x_1) F_{t,0,h,g}^{*(1)}(x_2). \end{aligned}$$

We calculate  $F_{t,0,h,g}^{*(1)}(x)$  for  $[0, 5]$  with span  $10^{-4}$  by using the Gauss-Kronrod method and Mathematica.

In the lower Figures in 1-3 in [17], we calculate (34) by using the above  $F_{0,h,g}^*(x)$  and by the trapezoidal rule with span  $10^{-4}$ .

In Figures 4-6 in [17], we obtain the data for  $F_{t,0,h,g}^*(x)$  for  $t = 10^{-7}$ , but we can not write their graphs for their very high oscillations, however, we can calculate (34) similarly. From the Figures, we can realize that we can obtain the desired inversion formulas concretely and exactly. Roughly speaking, these calculations are done with 1/30 time for the case of the Sobolev spaces in [15] and the results obtained there are, already, almost perfect, as we see from Figure 6 in [17].

We obtain

$$(L_t M_{t,0,h} g)(x_1, x_2) = \frac{1}{\pi} \int_{\pi(-x_1-1)/h}^{\pi(-x_1+1)/h} \frac{\sin p}{p} dp \cdot \frac{1}{\pi} \int_{\pi(-x_2-1)/h}^{\pi(-x_2+1)/h} \frac{\sin q}{q} dq.$$

Then, we can use Mathematica for  $h = 1/300, 1/400, 1/500, 1/1000$ , and we can see that the results above using our inversion formulas coincide with those results with 7 significant digits. Therefore, we may say that our real inversion formulas and calculations are justified; that is, we were able to realize numerically the inversion formula for the Gaussian convolution by using computers.

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