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Merging conditional stability and profile functions for obtaining convergence rates of general linear regularization methods

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The talk is devoted to the analysis of *linear inverse problems* that can be described as operator equations $Ax = y$ with an injective forward operator $A : X \to Y$ mapping between Hilbert spaces X, Y. For the solution $x^\dagger \in X$ we observe noisy data $y^\delta \in Y$ with $\|y^\delta - Ax^\dagger\| \le \delta$ and we assume A to be compact which leads to the *ill-posedness* of the equation. To overcome the difficulties arising from the ill-posedness a general linear regularization scheme is exploited based on bounded and piecewise continuous functions

$$g_\alpha(t) \quad (0 < t \le a := \|A^*A\| = \|A\|^2)$$

defined for regularization parameters $0 < \alpha \le \overline{\alpha}$. The family of functions g_α with

$$\lim_{\alpha\to 0} g_\alpha(t) \;=\; \frac{1}{t} \quad (0 < t \le a) \tag{1}$$

determines the regularization method with regularized solutions

$$x_\alpha^\delta \;=\; g_\alpha(A^*A)A^*y^\delta\,. \tag{2}$$

Moreover we assume the existence of a constant $\gamma_0 > 0$ such that

$$\sup_{0<t\le a} |\, t\, g_\alpha(t)\,| \;\le\; \gamma_0 \quad (0 < \alpha < \overline{\alpha})\,. \tag{3}$$

We say that $f(t)$ $(0 \le t \le \bar{t})$ is an *index function* if f is a real continuous and strictly increasing function with $f(0) = 0$. Following the concept of [2] such index functions are *profile functions* for the regularization method under consideration if they turn out to be majorant functions of the noise-free regularization error depending on the regularization parameter α. The generalized definition (see [5]) of the *qualification* of a regularization method expressed by a function is useful in this context. An index function $\psi(t)$ $(0 \le t \le a)$ is said to be a qualification with constant $1 \le \gamma < \infty$ for the regularization method g_α applied to the operator equation if for some $\tilde{\alpha} \in (0, a]$

$$\sup_{0<t\le a} |r_\alpha(t)|\, \psi(t) \;\le\; \gamma\, \psi(\alpha) \qquad (0 < \alpha \le \tilde{\alpha})\,, \tag{4}$$

where $r_\alpha(t) := 1 - t\, g_\alpha(t)$ $(0 < t \le a)$ denotes the corresponding bias function of the regularization method.

Along the lines of [2] we gain results for the decay of profile functions with the help of general source conditions $x^\dagger = \psi(A^*A)w$ $(w \in X)$. Here as in [3] we can show the impact of *conditional stability* by introducing an appropriate conditional stability function. An index function $\beta(\delta)$ $(0 \le \delta \le \bar{\delta})$ is said to be a conditional stability function for the set $M \subset X$ and the operator equation $Ax = y$ if

$$\sup_{y_1,y_2\in A(M):\, \|y_1-y_2\|\le\delta} \|A^{-1}y_1 - A^{-1}y_2\| \le \beta(\delta)\,, \tag{5}$$

where we set $A(M) := \{y \in Y : y = Ax \text{ for some } x \in M\}$.

Based on functions β now we can combine the ideas concerning the conditional stability on some subset $M \subset X$ and the ideas concerning the qualification of a regularization method in order to find convergence rates for general linear regularization methods. Thus, general source conditions are not needed. As an immediate consequence one can see that Hölder convergence rates and logarithmic convergence rates appear under certain conditions on the conditional stability function.

The method *Landweber iteration* will be taken into consideration as an example. A special treatment for the case of a self-adjoint forward operator A and the application of *Lavrentiev's regularization method* (see [4] and [6]) are presented as well. Further details will be given in the forthcoming paper [1].

References

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