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On inverse scattering on graph

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Differential operators on graphs often appear in mathematics, mechanics, physics, geophysics, physical chemistry, biology, electronics, nanoscale technology (for more details see, for example, [1] ,[2]). Scattering problem on graphs are of great importance for application. Inverse scattering problems were investigated, e.g., in: [3],[4], [5], [6]

We are interested mostly in inverse scattering problems on graphs, containing cycles.

We denote by Γ a graph, consisting of the "semi-infinite string" $\gamma = \{0 < x_1 < \infty\}$ and the "loop" $\kappa = \{0 < x_2 < 2\pi\}$, which are joint at the point $\{x_1 = 0\} = \{x_2 = 0\} = \{x_2 = 2\pi\}$, which we call later the "vertex" of graph Γ. The space $L_2(\Gamma)$ consists of functions $f(x)$ that are defined and square integrable on "semi-infinite string" γ and "loop" κ and we define $\|f\|^2_{L_2(\Gamma)} = \|f\|^2_{L_2(\gamma)} + \|f\|^2_{L_2(\kappa)}$.

Let us define differential equation

$$l(y) = \lambda^2 y(x),\ x \in \Gamma. \tag{1}$$

where

$$l(y) = -y''(x) + q(x)y(x),\ x \in \Gamma.$$

Here differentiation with respect to variable x is understood as differentiation with respect to variable x_1, when $x \in \gamma$ and as differentiation with respect to variable x_2, when $x \in \kappa$. (Differentiation is not defined at vertex $\{x_1 = 0\} = \{x_2 = 0\} = \{x_2 = 2\pi\}$).

The following boundary condition at vertex of graph Γ is called Kirchhoff boundary condition : function $y(x)$ is continuous at the vertex:

$$y(x_1 = 0) = y(x_2 = 0) = y(x_2 = 2\pi) \tag{2}$$

and sum of outgoing derivatives of function $y(x)$ at the vertex of graph Γ is equal to zero, i.e., derivatives with respect to local variables satisfy the equation

$$y'(x_1 = 0+0) + y'(x_2 = 0+0) - y'(x_2 = 2\pi - 0) = 0 \tag{3}$$

We suppose later on that function $q(x)$ is real valued, $(1+x_1)q(x_1) \in L^1(0,\infty)$ and $q(x) \in L^2_{loc}(\Gamma)$. Under such conditions operator L,which acts on Γ as $l(y)$: $Ly = l(y)$, with domain of definition

$$D_L = \{y \in W_2^2(\gamma \cup \kappa) : y \text{ satisfies boundary condition}(2)-(3)\}, \tag{4}$$

is self adjoint operator in the space $L_2(\Gamma)$.

As result, eigenvalues of operator L are real.

Let us mention now, that eigenvalues of operator L can be devided in two types: "proper" and "improper" eigenvalues.

Definition 1. Eigenvalue λ^2 is called a "proper eigenvalue" of operator L, if equation $l(y) = \lambda^2 y$ posseses nontrivial solution $y(\lambda, x) \in L^2(\Gamma)$, satisfying boundary conditions (2)–(3), such that it is nontrivial on semiinfinite string γ:

$$y(\tau, x_1) \not\equiv 0,\ x_1 \in \gamma. \tag{5}$$

Otherwise eigenvalue of problem (1)–(3) is called "improper eigenvalues".

Operator L posesses only finite number of proper eigenvalues $(i\lambda_1)^2, \ldots, (i\lambda_n)^2$, which are all simple and strictly negative: $0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n$.

Let us stress here, that operator L can have infinitely many improper eigenvalues, for example, in the case $q(x) \equiv 0,\ x \in \Gamma$, any point $\tau_n = (2n)^2,\ n = 1, 2, \ldots$ is a "improper eigenvalue" of operator L.

There exists unique function $S(\lambda)$, which is defined and continuous for any real $\lambda \neq 0$, bounded in any neighbourhood of point $\lambda = 0$ and such that there exists solution to equation (1), satisfying boundary conditions (2)–(3), which coincides on "string" γ with function

$$\phi(\lambda, x_1) = e(-\lambda, x_1) - S(\lambda) e(\lambda, x_1),\ x_1 \in \gamma. \tag{6}$$

Here $e(\lambda, x_1)$ is the Jost solution of equation $l(y) = \lambda^2 y$ on the semiaxis γ, which can be represented (see ([7])) for any λ, such that $\Im\lambda \geq 0$, in the form

$$e(\lambda, x_1) = e^{i\lambda x_1} + \int_{x_1}^{\infty} K(x_1, t) e^{i\lambda t} dt, \tag{7}$$

where $K(x_1, t) \in W_2^1(0 < x_1 < t < \infty)$ and

$$K(x_1, x_1) = \frac{1}{2} \int_{x_1}^{\infty} q(t) dt. \tag{8}$$

Definition 2. We call function $S(\lambda)$ the "scattering function".

Let us denote

$$m_j = \|E(i\lambda_j, \cdot)\|_{L_2(\Gamma)}^{-1}, \tag{9}$$

where $E(i\lambda_j, x)$ is eigenfunction of operator L, corresponding to proper eigenvalue $(i\lambda_j)^2$, normalized by initial condition $E(i\lambda_j, 0) = e(i\lambda_j, 0)$, i.e $E(i\lambda_j, x_1) = e(i\lambda_j, x_1),\ x_1 \in \gamma$.

Definition 3. We call $m_j,\ j = 1, \ldots, n$ the "weight numbers" of problem (1)-(3).

Problem. We investigate the problem of reconstruction of potential $q(x)$ by the means of "scattering and spectral data" $\{S(\lambda), \lambda \in \Re,\ \lambda_j,\ m_j,\ j = 1, ..., n\}$.

We denote

$$F(x) = \sum_{k=1}^{n} m_k^2 e^{-\lambda_k x} + F_S(x), \tag{10}$$

$$F_S(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (S_0(\lambda) - S(\lambda)) e^{i\lambda x} d\lambda. \tag{11}$$

Here $S_0(\lambda)$ is scattering function of problem (1)–(3) with $q(x) \equiv 0,\ x \in \Gamma$, i.e.

$$S_0(\lambda) = \frac{2\sin\lambda\pi - i\cos\lambda\pi}{2\sin\lambda\pi + i\cos\lambda\pi} \tag{12}$$

Main Theorem Let us suppose $q(x) > 0,\ x \in \kappa$ Then for any fixed $x \in \gamma$ the kernel $K(x, t)$ of transformation operator (7) satisfies an equation

$$F(x+t) + K(x,t) + \int_{x}^{\infty} K(x,y) F(t+y) dy = 0, \quad 0 < x < t < \infty. \tag{13}$$

It can be shown, that this equation is uniquely solvable and we can reconstruct potential $q(x_1),\ x_1 \in \gamma$

References

[1] P. Kuchment, Graph models of wave propagation in thin stuctures, *Waves random Media*, **12** (2002), R1–24

[2] Yu.V. Pokornyi, O.M. Penkin, V.I. Pryadiev, A.V. Borovskikh, K.P. Lazarev and S.A. Shabrov, Differential equations on geometric graphs (in Russian), *Fiziko-Matematicheskaya Literatura*, *Moscow*, (2005)

[3] N.I Gerasimenko, Inverse scattering problem on a noncompact graphs, *Teor. Mat Fiz.*, **75** (1988) pp.187–200 (Engl. transl. *Theor Math Phys.* **75** (1988) pp.230-240)

[4] M. Harmer, Inverse scattering for the matrix Schrodinger operator and Schrodinger operator on graphs with general self-adjoint boundary conditions, *ANZIAM J.*, **44**, (2002), pp. 161–168

[5] P. Kurasov and F. Stenberg, On the inverse scattering problem on branching graphs, *J. Phys A* **35** (2002) pp. 101–121

[6] J. Boman and P. Kurasov, Symmetries of quantum graphs and the inverse scattering problem, *Advances in Applied Mathematics* **32** (2005) pp. 58–70

[7] V.A. Marchenko, Sturm-Liouville Operators and Applications, *Operator Theory: Advances and Applications, Birkhauser* **22** (1986)

[8] Z.S. Agranovich and V.A. Marchenko, The Inverse Problem of Scattering Problem, *Gordon and Breach, New York and london* (1963)