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A pseudoparabolic problem of Barenblatt type with constraint in stratigraphy

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1 Introduction and mathematical formulation

In this presentation, one is interested in studying a mathematical problem arising from the modelling of geological basin formation. The initial model has been developed by the Institut Français du Pétrole and it takes into account sedimentation, transport and accumulation, erosion phenomena at large scales in time and space. The main feature of these models is characterized by a constraint on the time-derivative of the solution. This constraint leads us to consider an original class of conservation laws of degenerated pseudoparabolic type, close to the equation of Barenblatt that occurs in the theory of elastic fluids in elasto-plastic porous media (*cf.* G.I. Barenblatt [5]).

Concerning physical and numerical descriptions of these models for the multilithological case, one may find information in R. Eymard *et al.* [9, 10] and V. Gervais *et al.* [15] .

Concerning the monolithological case, one would find information on physical, numerical and mathematical descriptions in S. N. Antontsev *et al.* [3]-[4], J. Blum *et al.* [6], R. Eymard et al. [7, 8], G. Gagneux *et al.* [12, 11] and G. Vallet [16].

For a brief presentation of the model: let us consider a sedimentary basin whose base, denoted by Ω, is a fixed bounded domain of $\mathbb{R}^d$ $(d = 1, 2)$ with a Lipschitz boundary Γ.

Governing equations:
The sediment height, denoted by u, naturally satisfies the mass balance equation

$$\partial_t u + div\,\{\vec{q}\} = 0 \text{ in }]0, T[\times\Omega, \tag{1}$$

where the flux $\vec{q}$ is given by the empirical relation

$$\vec{q} = -\lambda(u)\mu\nabla[u + \tau\partial_t u] \text{ (μ is playing here the role of a limiter defined later)}, \tag{2}$$

according to a Darcy-Barenblatt’s law.

In the framework of a perfect physical equilibrium, the flux is considered to be proportional to the slope by writing that $\vec{q} = -\lambda\mu\vec{S}$ where $\vec{S} = \nabla u$, *i.e.* $\tau = 0$. Then, one gets a degenerated hyperbolic-parabolic problem (see S. N. Antontsev *et al.* [1, 4] for example).

But, taking into account a balancing of the slope leads us to consider $\vec{q} = -\lambda\mu\vec{S_e}$ with the first order kinetic equation $\vec{S_e} = \vec{S} + \tau\partial_t\vec{S}$, where $\tau > 0$ is time-scaled, and one gets a pseudoparabolic type equation (see S. N. Antontsev *et al.* [2] and G. Gagneux *et al.* [13]).

On the one hand, the diffusive coefficient λ depends on the depositional environment and is modeled as a nonlinear function of u.

On the other hand, in a sedimentary basin formation process, sediments must first be produced *in situ* by weathering effects prior to be transported by surfacing erosion. Thus, R. Eymard *et al.* [14] introduce a maximum erosion rate E such that $-\partial_t u \leq E$ in $]0,T[\times\Omega$, where E is nonnegative and takes into account the composition, the structure and the age of the sediments. Then, the authors propose a sediment flux given by $\overrightarrow{q} = -\lambda(u)\mu\,\nabla[u+\tau\partial_t u]$ where the dimensionless quantity μ, $0 \leq \mu \leq 1$, is playing the role of a flux limiter, with the relevant law of state in the form:

$$1-\mu \geq 0,\ \partial_t u + E \geq 0,\ (1-\mu)\,(\partial_t u + E) = 0 \text{ a.e. in }]0,T[\times\Omega. \tag{3}$$

To our knowledge, such a model appears as a non standard free boundary problem and does not admit a variational inequality formulation in order to describe these unilateral conditions.

Note, following S. N. Antontsev *et al.* [4], G. Gagneux *et al.* [11] and G. Vallet [16], that for natural solutions u in $L^2(0,T;H_0^1(\Omega))$ with $\partial_t u$ in $L^2(0,T;H_0^1(\Omega))$, (1, 2, 3) is equivalent to the following conservative formulation:

$$\begin{aligned} & 0 \in \partial_t u - div\{H(\partial_t u + E)\lambda(u)\nabla[u+\tau\partial_t u]\} \text{ in }]0,T[\times\Omega, \\ i.e. \qquad & 0 = \partial_t u - div\{\mu\lambda(u)\nabla[u+\tau\partial_t u]\} \text{ where } \mu \in H(\partial_t u + E), \end{aligned} \tag{4}$$

with the boundary conditions: $u = \partial_t u = 0$ on $]0,T[\times\Gamma$, the initial condition $u(0,.) = u_0$ with $u_0 \in H_0^1(\Omega)$ and where H denotes the maximal monotone graph of the function of Heaviside.

Moreover, the structural condition $\mu \in H(\partial_t u + E)$ leads to the following simplified formulation,

$$0 \quad = \quad \partial_t u - div[\mu\lambda(u)\nabla u] - \tau div[\lambda(u)\nabla(\partial_t u)],$$

according to the chain rule, valid for the weak derivatives of Sobolev functions.

Existence and uniqueness results for a solution to the above differential inclusion are still open ill-posed problems. Our purpose is to analyze a modified model where H is replaced by a continuous function a, an approximation Yosida of H for example.

Mathematical formulation:

Let us consider a Lipschitz bounded domain Ω with boundary Γ . For any positive T, let us denote $Q :=]0,T[\times\Omega$.

Therefore, we are interested in the following pseudoparabolic problem **(P)**:

$$\partial_t u - div[\lambda(u)a(\partial_t u + E)\nabla u] - \tau div[\lambda(u)\nabla\partial_t u] = 0 \quad in \quad Q, \tag{5}$$

with the initial height given by: $u(0,x) = u_0(x)$, $x \in \Omega$, where $u_0 \in H_0^1(\Omega)$
and homogeneous Dirichlet boundary conditions for u and $\partial_t u$.
In the sequel, one assumes **(H)**:

$\tau > 0$, $E : [0,T] \to \mathbb{R}$ is a nonnegative continuous function, a is a θ-Hölderian continuous function (with $\theta \geq \frac{1}{2}$) defined on $\mathbb{R}$ that satisfies

$$0 \leq a \leq a_{\max}, \quad a(0) = 0, \quad \forall x > 0,\ a(x) > 0 \text{ and } a(-x) = 0 \text{ by extension},$$

and λ is a continuous function such that $\exists \lambda_0, \lambda_{\max} \in \mathbb{R}, \quad 0 < \lambda_0 \leq \lambda \leq \lambda_{\max}$.

Provided that existence results are proved, the definition and basic properties of a solution to the problem (P) are developed.

By considering weak solutions, one would consider that

Definition 1 *A solution to Problem (P) is any u in $H^1(0,T;H_0^1(\Omega))$, such that for any v in $H_0^1(\Omega)$ and for a.e. t in $]0,T[$,*

$$\int_\Omega \{\partial_t u v + \lambda(u)a(\partial_t u + E)\nabla u.\nabla v + \tau\lambda(u)\nabla\partial_t u.\nabla v\}\,dx = 0. \tag{6}$$

with the initial condition $u_{|t=0} = u_0$ a.e. in Ω.

2 Existence of a solution and smoothing properties

This section provides the following existence and regularity result, thanks to the classical theorems of N. G. Meyers and J. Nečas:

Theorem 1 *Assume (H). Then, for any u_0 in $H_0^1(\Omega)$, a solution u to problem (P) exists in the sense of definition 1*

If moreover, $u_0 \in W_0^{1,p_0}(\Omega)$, with a given $p_0 > 2$, then one has that $u \in W^{1,\infty}(0,T; W_0^{1,p_0}(\Omega))$.

Next, let us note that this regularity can be obtained for a larger class of values of p.

In order to, consider the function $v = \partial_t u$ satisfying

$$v - \tau div[\lambda(u)\nabla v] = F = div[\overrightarrow{G}] \quad in \quad Q,$$

with

$$\overrightarrow{G} = \lambda(u)a(\partial_t u + E)\left(\nabla\int_0^t vds + \nabla u_0\right).$$

It follows from the regularity $u \in W^{1,\infty}(0,T; W_0^{1,p_0}(\Omega))$ the estimates

$$|u(x,t) - u(y,t)| \leq C|x-y|^{\alpha},\ 0 < \alpha \leq \frac{p_0 - 2}{p_0} < 1,\ d = 1, 2,\ p_0 > 2,$$

$$|\lambda(x,t) - \lambda(y,t)| = |\lambda(u(x,t)) - \lambda(u(y,t))| \leq C|x-y|^{\alpha},\ 0 < \alpha \leq \frac{p_0 - 2}{p_0} < 1. \tag{7}$$

Next we use the following standard way. We introduce

$$v_k = v(x,t)\chi_k(x), k = 1, ..., n,\ \sum_{k=1}^{n} \chi_k(x) = 1,$$

$$\chi_k \in C^2(B_{k,\rho}),\ B_{k,\rho} = \{x : |x - x_k| < \rho\}, \chi_k|_{\partial B_{k,\rho}} = 0,\ and\ 1 \leq \chi_k(x), if\ x \in B_{k,\frac{\rho}{2}}$$

The function v_k satisfies

$$v_k - \tau div[\lambda(u)\nabla v_k] = F^* := div[\lambda(u)a(\partial_t u + E)\left(\int_0^t \nabla v_k ds\right)] + g^*,\ in \quad B_{k,\rho}.$$

Applying (7) and assuming $\rho \leq \rho_0$ (small enough), we obtain the estimate

$$\|v_k\|_{W_0^{1,p}} \leq C(p)\left[a_{\max}\lambda_{\max}\left\|\int_0^t \nabla v_k ds\right\|_{L^p} + \|g^*\|_{L^q}\right],\ q = \frac{pd}{pd - p + d}\ ,\ k = 1, ..., n,$$

for any finite $p > 2$. Last estimate and Gronwall's inequality lead us to the estimate

$$\|v\|_{W_0^{1,p}(\Omega)} \leq C,$$

for any finite $p > 2$.

3 Uniqueness of the solution

Then, we are interested in the uniqueness of such a solution, for τ large enough or for u_0 small enough (in the sense of $W_0^{1,p}(\Omega)$). Indeed, for a given topography u_0, there is a critical value τ^* for τ above which the solution is unique.

In the sequel, one would consider hypothesis **(H')**:
with (H), Lipschitz-continuous assumption for the state functions a and λ would be considered. Let us denote by $\text{lip}_a = ||a'||_\infty$ and $\text{lip}_\lambda = ||\lambda'||_\infty$ respectively.

The aim of this section is to claim that the following properties hold:

Proposition 1 *Assume (H') and that the solution u of the previous section satisfies :*
$\tau > 2\frac{\lambda_\infty}{\lambda_0} lip_a ||u||_{W^{1,\infty}(0,T;W^{1,p}(\Omega))}$,
then the solution to the problem (P) is unique.

Moreover, the application $u_0 \in H_0^1(\Omega) \mapsto \partial_t u \in L^\infty\big(0,T;H_0^1(\Omega)\big)$ *is a locally Lipschitz-continuous function.*

Remark:
Since a positive constant $C = C(||u_0||_{W^{1,p}(\Omega)})$ exists such that $||u||_{W^{1,\infty}(0,T;W^{1,p}(\Omega))} \leq C$, it comes that a nonnegative real $\tau^* = \tau^*(||u_0||_{W^{1,p}(\Omega)})$ exists such that, for any $\tau > \tau^*$, a unique solution u to problem (P) exists in the sense of definition 1.

It should be mentioned that the well-posedness in case $\tau = 0$ is a challenging problem (see G. Vallet [16] for $d = 1$).

Another challenging problem is to find other function spaces for the well-posedness without smallness condition on the initial data, for any positive τ or to establish the uniqueness criteria that are as general as possible.

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