CRYSTALLOGRAPHIC AND QUASICRYSTALLOGRAPHIC GROUPS

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Содержание

1 A SHORT HISTORICAL REVIEW...

1.1 History

1890–91 years: E.S. Fedorov and A. Schönflies classified all possible symmetry groups of natural crystalls or *crystallographic groups.*

A crystall symmetry is a transformation (isometry) of a euclidean plane, which moves the atomic lattice of the crystall to itself. It turned out that the number of such groups and, consequently, different atomic lattices is equal to **219**.

If we distinguish a lattice from its mirror reflection then this number equals **230**. Crystall always admits parallel shifts (translations) in three non-coplanar directions. There are only finitely many of rotation symmetries modulo translation subgroup.

Year 1900: D. Gilbert formulated his 18-th problem on filling the space using congruent polyhedra.

The first part of the problem is the question: are there only finitely many essentially different groups posessing a fundamental region (crystallographic groups) in the n-dimensional euclidean space?

The second part asks if there exist any polyhedra completely filling the space without gaps and which are not fundamental regions for any groups of motions.

Years 1911–1912: L. Bieberbach answered the first part of Gilbert problem in the affirmative: the translation subgroup of a crystallographic group has the finite index, this index is bounded by Jordan theorem (1878, 1880), there are only finitely many of types of isomorphisms.

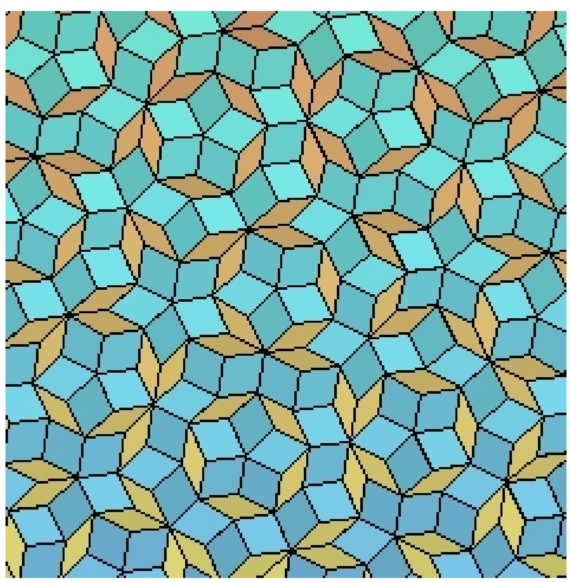
Year 1928: K. Reinhardt answered the second part of the 18-th Gilbert problem in the affirmative: such polyhedra exist.

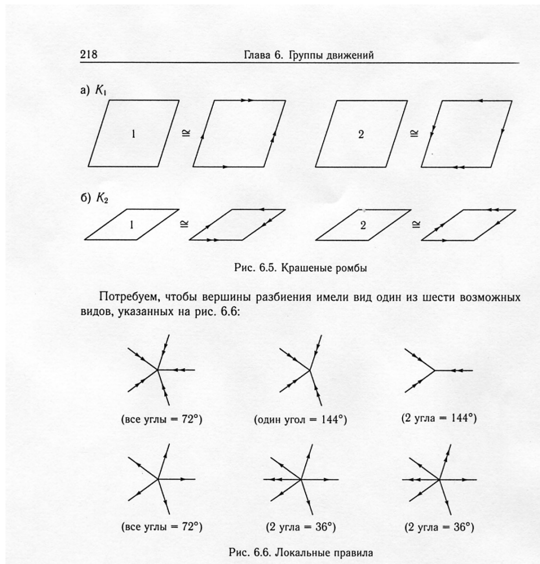
Year 1947: H. Zassenhaus offered an algorithm for encountering crystallographic groups and, together with his pupils, found the number of crystallographic groups in 4-dimensional euclidean space. It turned out that this number equals **4783**.

Year 1961: B.N. Delaunay (Delone) proved that there are only finitely many of types of such polyhedra, which are Dirichlet regions for some crystallographic groups in the n-dimensional euclidean space. He offered an algorithm for encountering such polyhedra for given n.

Years 1960–1975: Hao Wang, R. Berger, D.E. Knuth, R.M. Robinson, R. Ammann, ... there were discovered many examples of aperiodic tilings ("aperiodic" means "without any nontrivial translations along itself") of the plane by "bricks" of finitely many types. The corresponding groups of motions, moving the tiling to itself, turned out to be trivial.

Year 1978: R. Penrose there were discovered aperiodic Penrose tilings.





218 Глава 6. Группы движений

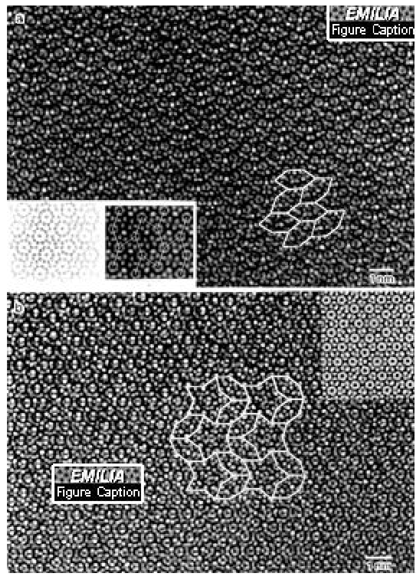
Рис. 6.5. Крашеные ромбы

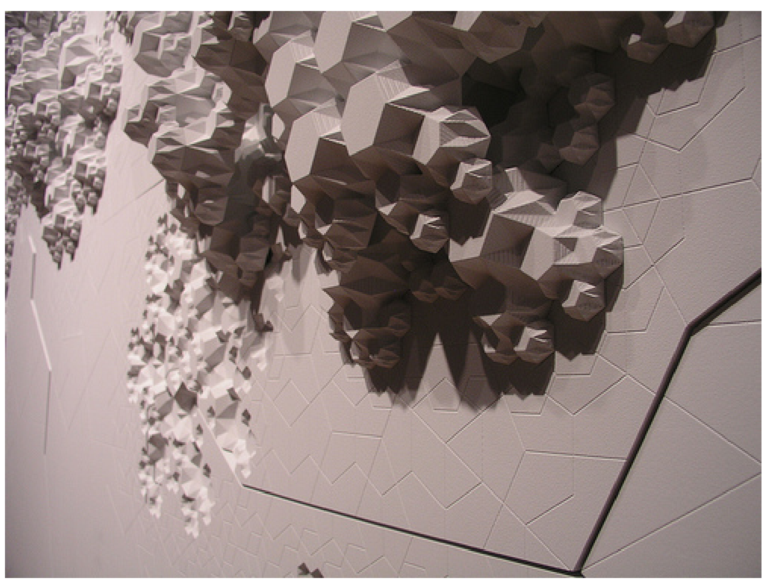
Потребуем, чтобы вершины разбиения имели вид один из шести возможных видов, указанных на рис. 6.6:

Рис. 6.6. Локальные правила

Year 1981: de Bruijn found the "cut and project method" for Penrose tilings. He obtained Penrose tilings by projecting 2-dimensional faces of a "cubic" neighbourhood of some appropriate plane in the cubic tiling of the 5-dimensional space to this plane.

Year 1984: D. Schechtman and others there were discovered really existing quasicrystalls with the atomic lattice of a new type. There were experimentally discovered rotation symmetries of orders **5, 8, 10, 12**, which is impossible in Fedorov–Schöenflies groups. It is possible to obtain Penrose tilings by intersecting a quasicrystall with a plane.





Year 1986: S.P. Novikov and A.P. Veselov introduced the notion of a quasicrystallographic group containing a translation quasilattice.

The translation quasilattice is finitely generated and contains a basis of the space.

Main problem Found a classification of quasicrystallographic groups in dimension 2 and 3.

Years 1990–92: there were investigated flat quasicrystallographic groups by Novikov's school (S.A. Piunikhin, A.P. Veselov, A. Lazareva, A. Chalych, V.Roganova).Theory of such groups with quadratic quasilattices provides rotations of finite orders **5, 8, 10, 12**.

The possibility of rotation symmetries of an infinite order was discovered. There were investigated icosahedral models of quasicrystalls which admit physical realizations.

There was found a connection between quasicrystallographic groups and crystallographic groups in pseudo-euclidean spaces.

G.A. Margulis constructed examples of wild quasicrystallographic groups in the 3-dimensional space.

It is possible that a rotation group is everywhere dense in the group SO_3 of all rotations and, at the same time, the whole group is not contained in a finitely generated quasicrystallographic group.

Years 2000–2003: R.M. Garipov found a new "algebraic method of encountering crystallographical groups".

Algorithm is based on the weakened Bieberbach theorem and allows to compute not only classical crystallographic groups, but also crystallographic groups of motions of Minkowsky spaces with a fixed rotation group. There were performed corresponding computations for the Minkowsky spaces of dimension $\leqslant 4$.

But the problem of extending the weakened Bieberbach theorem to arbitrary pseudo-euclidean spaces wasn't solved.

1998-2008 year: R.V. Moody formulated the general definition of a quasicrystall in terms of cut and projection.

There were showed outreachs to geometry, discrete groups and arithmetics, analysis and dynamic systems theory.

2006-10 years: V.A. Artamonov and S. Sanchez introduced the notion of the symmetry group of a quasicrystall in "cut and projection model"terms.

Numbers 5, 8, 10, 12 appear again.

V.A. Churkin solved the problem of extending the weakened Bieberbach theorem to arbitrary pseudo-euclidean spaces.

1.2 A short plan of lectures

- Main definitions and formulas for affine transformations and groups of motions.

  Different definitions of crystalls and crystallographic groups.

- The weakened Bieberbach theorem in euclidean spaces.
- The Jordan theorem.
- A short proof of the Bieberbach theorem on the finiteness of the number of isomorphism classes of n-dimensional crystallographic groups.
- Auxiliary information about algebraic integers.
- Characterization of rotation angles for 2-dimensional and 3-dimensional quasicrystallographic groups and rotation angles for groups with quadratic quasilattices.
- Wild examples in dimensions 4 and 3.
- Pseudo-orthogonality of the rotation group of a quasicrystallographic group.
- The weakened Bieberbach theorem in pseudo-euclidean spaces of index $\leqslant 2$.
- Examples of crystallographic groups of motions of pseudo-euclidean spaces of index $\geqslant 3$, that contain several "lattices and also exceptional automorphisms, which permute the translation lattice with another abstract "lattices".
- Unsolved problems.

1.3 Advisable sources

- *J.G. Ratcliff,* Hyperbolic manifolds, Springer-Verlag, 1994.

  There is a detailed elementary proof of Bieberbach theorem and the theory of discrete cocompact groups of motions of euclidean, spherical and hyperbolic spaces.

- *В.А. Артамонов, Ю.Л. Словохотов,* Группы и их применение в физике, химии и кристаллографии, Москва, Академия, 2005.

  This is a detailed and interesting textbook, intended for chemists, physisists and applied mathematicians.

- *Р.М. Гарипов,* Алгебраический метод вычисления кристаллографических групп и рентгеноструктурный анализ кристаллов, Новосибирск, НГУ, 2003.

  A textbook with Garipov's algorithm and it's realization in the 3-dimensional space.

- *С.П. Новиков, И.А. Тайманов,* Современные геометрические структуры и поля, Москва, МЦНМО, 2005.

  There is an introduction to crystallographic groups theory and it's generalizations.

- *Ле Ты Куок Тханг, С.А. Пиунихин, В.А. Садов,* Геометрия квазикристаллов, Успехи математических наук, 1993, Т. 48, N 1, С. 41–102.

  There are results of Novikov's school on the quasicrystallographic groups.

- *И.Р. Шафаревич,* Основные понятия алгебры, Ижевск, РХД, 1999.

  A beautiful outline of the classical theory of crystallographic groups (without proofs).

- *R.V. Moody,* Model sets: a survey, From Quasicrystals to More Complex Systems, Springer-Verlag, Berlin, 1998, p.145–166. (Variant 2008 year: arXiv: math.MG/0002020.)

  General definition of quasicrystall in the "cut and projection"model: review of results and emphasizing on relationships with geometry, discrete groups and arithmetics, analysis and dynamic systems theory.

- *R.V. Moody,* Model sets: a survey, From Quasicrystals to More Complex Systems, Springer-Verlag, Berlin, 1998, p.145–166. (Variant 2008 year: arXiv: math.MG/0002020.)

General definition of quasicrystall in the "cut and projection"model: review of results and emphasizing on relationships with geometry, discrete groups and arithmetics, analysis and dynamic systems theory.

2 AFFINE TRANSFORMATIONS

2.1 Definitions

Let us consider $\mathbb{R}^n$ as a vector space and as an affine space simultaneously. Elements of the vector space $\mathbb{R}^n$ are vectors and there is a special (distinguished) vector 0, automorphisms of this vector space are invertible linear transormations $x \mapsto Ax, \quad \det A \neq 0$.

The affine space $\mathbb{R}^n$ has no distinguished elements, its elements are called 'points' and automorphisms are invertible affine transformations $g : x \mapsto a + Ax$, where $a \in \mathbb{R}^n$, $\det A \neq 0$.

A linear transormation A is called *'the linear part'* of an affine transformation g or *'the differential' dg* of this transformation.

A map $x \mapsto a + x$ is called *'a parallel translation'* by the vector a.

It is convenient to write $g := a+A$. Here $+$ is a formal notation, but in the formula $g(x) = a+Ax$ the symbol $+$ means, as usually, the addition of vectors. Translation has the form $a + I$ where I is the identity map. Translation is defined uniquely by the corresponding vector and so it may be identifyed with this vector.

2.2 Properties of compositions

Theorem

1) $g : x \mapsto a + Ax$ determines a and A uniquely.

2) The composition law: if $g = a+A$, $h = b+B$ then $g \cdot h = a+Ab+AB$.

3) The inversion law: if $g = a + A$ then $g^{-1} = -A^{-1}a + A^{-1}$.

4) The conjugation law: if $g = a + A$ is an affine transformation and $z = z + I$ is a translation then $g \cdot z \cdot g^{-1} = Az$ is a translation.

Proof. 1) If $a + Ax = b + Bx$ for all x then for $x = 0$ we obtain $a = b$ and $Ax = Bx$ for all x, $A = B$.

2) $g(h(x)) = a + A(b + Bx) = a + Ab + ABx$.

3) If $g \cdot h = 0 + I$ then as a consequence of 2) and 1) we obtain $a + Ab = 0$, $AB = I$, thus $b = -A^{-1}a$, $B = A^{-1}$.

4) $g \cdot z \cdot g^{-1} = (a+A)(z+I)(-A^{-1}a+A^{-1}) = \quad = (a+Az+A)(-A^{-1}a+A^{-1}) = a + Az - a + I = Az + I = Az$.

The proof of the theorem is complete.

Corollary 1) The set Aff_n of all affine transformations $\mathbb{R}^n$ is a group under the operation of composition.

2) The map $g \mapsto dg$ is a group homomorphism $Aff_n \to GL_n$.

3) The subgroup T_n, consisting of all translations, is normal in Aff_n.

4) Let $\Gamma \leqslant Aff_n$ and $Z = \Gamma \cap T_n$. Then for all $g = a + A \in \Gamma$ and $z \in Z$ we have $Az \in Z$.

2.3 Pseudo-orthogonal transformations

Definition A non-degenerate symmetric bilinear form $B(x, y)$ of the signature (p, q), defined on the vector space $\mathbb{R}^n$, determines on $\mathbb{R}^n$ the structure of the pseudoeuclidean space $\mathbb{R}^{p,\,q}$. In the canonical form the corresponding quadratic form looks like

$$x_1^2 + \ldots + x_p^2 - x_{p+1}^2 - \ldots - x_{p+q}^2,$$

where $p + q = n$. The space $\mathbb{R}^{n,\,0}$ is called 'a euclidean space', $\mathbb{R}^{n-1,1}$ is a Minkowsky space.

Definition *A linear* transformation preserving a form of the signature (p, q) is called *'pseudoorthogonal* of type $(p,\,q)$'. Such transformations constitute the pseudoorthogonal group $\mathrm{O}_{p,\,q}$. Affine transformations $g = a + A$, where $A \in \mathrm{O}_{p,\,q}$, are called *'motions' or 'isometries'* of the pseudoeuclidean space $\mathbb{R}^{p,\,q}$. They constitute the group of motions $\mathrm{Is}\,(\mathbb{R}^{p,\,q})$.

Remark In the euclidean space the group $\mathrm{Is}\,(\mathbb{R}^{n,\,0})$ indeed coincides with the set of mappings preserving the distance between any two points (an excercise). In general case a quadratic form of the signature $(p,\,q)$ determines only pseudometric but not a metric: it is possible that $B(x, x) = 0$ for $x \neq 0$.

The group corresponding to the the 4-dimensional Minkowsky space is called 'the Poincare group', the corresponding motions preserve the Maxwell's equations for the electromagnetic wave in vacuum.

3 CRYSTALLOGRAPHIC GROUPS: DEFINITIONS

There exist different definitions of crystallographic groups and their generalizations.

Let X be a locally compact metric space with the metric $|x-y|$.A subset $D \subset X$ is a *Delaunay set* or a *net* in Xif there exist such numbers $R > r > 0$ that

$$\forall a,\ b \in D:\ a \neq b \Rightarrow\ |a-b| > r,$$
$$\forall x \in X\ \exists a \in D:\ |x-a| < R.$$

Definition 1
The group Γ of isometries of a metric space X is called *crystallographic* if its orbit Γx is a Delaunay set in X.

'Crystall' is a union of a finite number of orbits of a crystallographic group Γ. Usually, in applications the crystalls under investigation are ones in euclidean spaces.

The following definition of the crystallographic group doesn't involve the notion of metric. It is sufficient to have the topological structure, which can be found, for example, in affine spaces.

Let Γ be a group of continuous transformations of a locally compact topological space X. Then Γ acts on X *properly discontinuously* if

$$\forall K \subset X: \quad K \text{ is compact } \Rightarrow \text{ set } \{g \in \Gamma \,|\, gK \cap K \neq \oslash\} \text{ is finite.}$$

A group Γ acts on X *cocompactly*

if

$$\exists K \subset X: \quad K \text{ is compact and } X = \bigcup_{g \in \Gamma} gK.$$

Definition 2 *Crystallographic group* is a properly discontinuous cocompact group of topological transformations.

For the groups of motions of euclidean spaces these two definitions of a crystallographical group are equivalent.

Because of Bieberbach theorem it is possible to give the third definition of a crystallographic group.

Definition 3

The group of motions of a euclidean space is *crystallographic* if all of its translations form a *lattice*, that is, a free abelian group which is generated by the basis of the space.

The hard part of the proof of Bieberbach theorem is how to deduce the third definitions from the first and the second definition.

The third definition can be easily generalized to the Novikov-Veselov definition of a *quasicrystallographic group of motions* of a euclidean space.

Definition 4

The group of motions of a euclidean space is *quasicrystallographic* if all of its translations form a *quasilattice*, that is, a finitely generated abelian group containing the basis of the space:

$$\Gamma < \mathrm{Is}\,(\mathbb{R}^{n,0}), \quad Z = \Gamma \cap T_n = \langle e_1, \ldots, e_N \rangle,$$

$$e_1, \ldots, e_n \; - \; \text{basis of } \mathbb{R}^{n,0}, \quad n \leqslant N.$$

Definition 5 Let bee Γ crystallographic or quasicrystallographic group with lattice Z of translations. The quotient group $G = \Gamma / Z$ is called *a group of rotations* or *a point symmetry group*. This relation is usually expressed by the formula $\Gamma = \Gamma(G, Z)$.

Further we will use the third definition of a crystallographic (and a quasicrystallographic) group in euclidean spaces. The third definition of a crystallographic group will also be used for pseudoeuclidean spaces.

Remark. The Auslender conjecture, stating that crystallographic groups of an affine space are virtually solvable, in general case still is not proved. It holds if the dimension of the space is $\leqslant 3$.

4 THE WEAK BIEBERBACH THEOREM

4.1 Proof

Theorem A lattice of translations in a crystallographic group of motions of a euclidean space is uniquely determined by the crystallographic group itself as an abstract group.

More exactly, it is

a) a normal free abelian subgroup of finite rank,

b) a maximal abelian subgroup,

c) endowed with a positive definite symmetric bilinear form which is invariant under the action of the group by conjugation.

Proof. Let Z be the lattice of translations of a crystallographic group Γ.

Let a subgroup $T \leqslant \Gamma$ satisfy the conditions a), b), c). Then
– $T \cap Z \triangleleft TZ$,
– $TZ/T \cap Z \simeq (T/T \cap Z) \times (Z/T \cap Z)$ is an abelian group,
– $T \cap Z$ is the center of TZ.

Hence, TZ is a nilpotent group of the class 2. Then

$$\forall g \in T,\ \forall z \in Z:\ [g,[g,z]] = 1.$$

If $g = a + A$ then

$$[g,z] = gzg^{-1}z^{-1} = A(z) - z = (A - I)z.$$

Then $[g,[g,z]] = (A-I)^2 z$. Therefore $(A-I)^2 z = 0, \quad \forall z \in Z$.

As Z contains a basis of $\mathbb{R}^n$ it follows that $(A-I)^2 = 0$. But $A \in \mathrm{O}_n(\mathbb{R})$. Hence $A - I$ is a normal transformation. If $(A-I)^2 = 0$ then $A - I = 0$, $A = I$ so g is a translation, $T \subset Z$. Due to the maximality $T = Z$.

The proof of the theorem is complete.

4.2 Corollary

Remark The proof does not work for pseudoeuclidean spaces because nontrivial unipotent transformation can be a motion of a pseudoeuclidean space.

But for quasicrystallographic groups it work.

Corollary Let $\varphi : \Gamma \to \Gamma'$ be an abstract isomorphism of crystallographic groups $\Gamma(G,Z)$ and $\Gamma'(G',Z')$. If $\varphi(Z) = Z'$ then after an appropriate conjugation in the group Aff_n we may assume that

$$G = G' \leqslant GL_n(\mathbb{Z}).$$

Proof. Let

$$\varphi : g = a + A \mapsto g' = a' + A', \quad z \mapsto f(z),\ z \in Z.$$

Here f is an isomorphism of abelian groups Z and Z' in $\mathbb{R}^n$. Then f can be represented by a linear transformation F of the space $\mathbb{R}^n$.

Due to the isomorphism

$$g \cdot z \cdot g^{-1} \mapsto g' \cdot f(z) \cdot g'^{-1}.$$

By the conjugation formula we have

$$A(z) \mapsto A'(f(z)),$$

$$f(A(z)) = A'(f(z)), \quad \forall z \in Z,$$
$$FA = A'F, \quad A' = FAF^{-1} = fAf^{-1}.$$

Replacing Γ with a group which is conjugate to Γ by f we can assume that $G = G'$. If we choose the basis of the space which is also a basis of the lattice Z, we get $G < GL_n(\mathbb{Z})$.

The proof of the corollary is complete.

5 FINITE SUBGROUPS OF $GL_n(\mathbb{Z})$

5.1 Jordan's Theorem

It is easy to see, that the orders of finite subgroups of $GL_n(\mathbb{Z})$ bounded by the function depending on n. Thus there are only finite number of finite subgroups up to isomorphism.

Indeed, ring homomorphism $\mathbb{Z} \to \mathbb{Z}_m$ by the rule $x \mapsto x(\operatorname{mod} m)$ induce group homomorphism $\varphi : GL_n(\mathbb{Z}) \to GL_n(\mathbb{Z}_m)$. If $m \geqslant 3$ then $\operatorname{Ker} \varphi = \Gamma_n(m)$ has no torsion (exercise).

Therefore any finite subgroup of $GL_n(\mathbb{Z})$ embedded in finite group $GL_n(\mathbb{Z}_3)$ and it's number not greater then the number of subgroups of $GL_n(\mathbb{Z}_3)$.

It is important to know the number of *conjugacy classes* of finite subgroups of $GL_n(\mathbb{Z})$ for the classification of crystallographic groups.

Theorem(Jordan)

There are only finitely many finite subgroups of $GL_n(\mathbb{Z})$ up to conjugacy in $GL_n(\mathbb{Z})$.

There are only a bounded number of possibilities for the action of finite group on a rank-n free abelian group Z.

5.2 Proof

Proof. The second statement follows from the first because the action of finite group G on $\mathbb{Z}^n$ is given by a homomorphism $\varphi : G \to GL_n(\mathbb{Z})$.

Two actions is isomorphic if and only if images $\operatorname{Im}\varphi$ of homomorphism is conjugated in $GL_n(\mathbb{Z})$.

We now prove the first statement.

Given a finite subgroup G of $GL_n(\mathbb{Z})$, we will find a basis for $\mathbb{Z}^n$ so that elements of G are represented by matrices with entries whose absolute values are bounded by a constant $C = C(n)$.

Consider the standard embedding $\mathbb{Z}^n \subset \mathbb{R}^n$, so that G can be seen as a subgroup of $GL_n(\mathbb{R})$.

By averraging over G any positive definite quadratic form on $\mathbb{R}^n$, find G-invariant positive definite quadratic form.

Scale so that the lattice points nearest the origin have length 1. Let $V \leqslant \mathbb{R}^n$ be the vector subspace spanned by these lattice points.

Scale the subspace $V^\perp$ until the lattice points in $\mathbb{R}^n \setminus V$ nearest the origin have length 1.

The scaled quadratic form is G-invariant because G preserves V and $V^\perp$.

Continue this way until the vector subspace spanned by lattice points of minimum (unit) norm is all of $\mathbb{R}^n$.

There may not be a basis for lattice consisting of elements of norm 1 (for example, union of set all cubes vertices and centers for cubical lattice in $\mathbb{R}^5$). We will show that there is a basis consist of elements of norm at most $(n+1)/2$.

Lemma 1. Number $N(r)$ of lattice points in ball with radius r and center at origin $\mathbb{R}^n$ is at most $(2r+1)^n$.

Proof. Indeed, balls with radius $1/2$ and centered at lattice points have no common points and contains in ball $B(0, r + (1/2))$.

We can using formula from math. analysis for volume of ball with radius r in $\mathbb{R}^n$:

$$\operatorname{vol} B^n(r) = c_n r^n,$$

constant $c_n = \operatorname{vol} S^{n-1}/n$ depends only on n, S^{n-1} is a sphere with unit radius from $\mathbb{R}^n$.

Lets compare volumes:

$$N(r) \cdot c_n/2^n \leqslant c_n(r + (1/2))^n, \quad N(r) \leqslant (2r + 1)^n.$$

Lemma 1 is proved.

Choose linearly independent elements $a_1, \ldots, a_n$ of norm 1 from $\mathbb{Z}^n$.

Let W_k be the vector space spanned by $a_1, \ldots, a_k$. Suppose inductively that for some $k < n$ the lattice $\mathbb{Z}^n \cap W_k$ has a basis $b_1, \ldots, b_k$ of vectors of norm $\leqslant (k + 1)/2$.

Lemma 2. The distance from $(\mathbb{Z}^n \cap W_{k+1}) \backslash W_k$ to W_k is greater than $(k + 3)^{-(k+1)}$.

Proof. Let exist $w = u + v$, $w \in \mathbb{Z}^n \cap W_{k+1}, u \in W_k$, $v \in W_k^{\perp}$, $0 < \|v\| \leqslant (k + 3)^{-(k+1)}$. Then the distance from the vectors λw, where $\lambda = 0, \pm 1, \pm 2, \ldots, \pm (k + 3)^{k+1}$, to space W_k is $\leqslant 1$.

Let P be a parallelepiped spanned by $a_1, \ldots, a_k$ with center at origin.

By adding to λw appropriate $\mathbb{Z}$-linear combination of elements $a_1, \ldots, a_k$ we can get such vector $(\lambda w)' \in \mathbb{Z}^n \cap W_{k+1}$ that it's projection to W_k belong to P.

Parallelepiped P has diameter not greater than $| a_1 | + \ldots + | a_k | = 1 + \ldots + 1 = k$. Thus P contained in a ball of radius $k/2$ with center at origin and $(\lambda w)'$ contained in a ball of radius $r = (k/2) + 1$ with center at origin.

The number of points $(\lambda w)'$ is not greater than $N(r)$, $r = (k/2) + 1$. Then by lemma 1 for a space $W_{k+1} = \mathbb{R}^{k+1}$ we have that

$$2(k + 3)^{k+1} + 1 \leqslant (2r + 1)^{k+1} = (k + 3)^{k+1}.$$

This inequality is false. Lemma 2 is proved.

A lattice is the additive group, then the infimum of lenghts projection of vectors $w \in \mathbb{Z}^n \cap W_{k+1}$ on $W_k^{\perp}$ is a minimum for some vector $w_0 \in \mathbb{Z}^n \cap W_{k+1}$ by lemma 2. Analogously, the projection of vector a_{k+1} is proportional to the projection of vector w_0 with a coefficient in $\mathbb{Z}$. Therefore $b_1, \ldots, b_k, w_0$ is a basis of lattice $\mathbb{Z}^n \cap W_{k+1}$. By adding to w_0 appropriate $\mathbb{Z}$-linear combination of elements $a_1, \ldots, a_k$ or $b_1, \ldots, b_k$ we can get such a new basis vector b_{k+1} that it's projection to W_k belong to P. Then

$$| b_{k+1} | \leqslant \sqrt{(k/2)^2 + 1} < (k + 2)/2.$$

This completes the inductive step.

We conclude by showing that matrices representing elements of G with respect to basis $b_1, \dots, b_n$ for lattice $\mathbb{Z}^n$ have bounded entries.

Since G preserves length, it sufficient to show that the coordinates of any vector v are bounded, if $| v | \leqslant (n+1)/2$.

By the Cramer's formulas i-th coordinate of vector v with respect to basis $\mathbb{Z}^n$ has absolute value $\operatorname{vol} P_i / \operatorname{vol} Q$, where P_i is a parallelepiped spanned by $b_1, \dots, b_{i-1}, v, b_{i+1} \dots, b_n$, and Q is a parallelepiped spanned by $b_1, \dots, b_{i-1}, b_i,\ b_{i+1}, \dots, b_n$.

Since volume of parallelepiped is not greater than product of lengths of it's edges, we have $\operatorname{vol} P_i \leqslant ((n+1)/2)^n$.

Now we only need a bottom estimation for $\operatorname{vol} Q$.
Open balls with radius

$$r_n = (n+3)^{-(n+1)}$$

and centered at the vertices of Q have no pairwise intersection.

It's intersections with Q give us a number of sectors, sum of volumes of which is not less then volume of Q.

Using parallel transfer we can collect all sectors at the same vertex of Q,

in that case they form a ball B with radius r_n and $\operatorname{vol} Q > \operatorname{vol} B(r_n)$.

As a result coordinates of vector v bounded by function

$$\operatorname{vol} P_i / \operatorname{vol} Q \leqslant ((n+1)/2)^n / (c_n r_n^n)$$

depending only on n.

The proof of the theorem is complete.

6 NUMBER OF CRYSTALLOGRAPHIC GROUPS UP TO ISOMORPHISM

6.1 Bieberbach's theorem

Theorem(Bieberbach)

There are only finitely many isomorphic classes of n-dimensional euclidian crystallographical groups for any natural n.

Proof. Let $\Gamma = \Gamma(G, Z)$ be a crystallographic group of motions of euclidian space $\mathbb{R}^n$ with translation lattice Z and point group G.

6.2 Cocycles and extensions

By the weakened Bieberbach theorem group isomorphism induces lattice isomorphism and point group isomorphism. By the Jordan's theorem we suppose that

- $Z = \mathbb{Z}^n$,
- G is fixed finite subgroup of $GL_n(\mathbb{Z})$,
- action G on $Z = \mathbb{Z}^n$ is given.

We will find the number of such groups $\Gamma \leqslant \mathrm{Is}\,(\mathbb{R}^n)$ that

- $Z \lhd \Gamma$,
- Z is a maximal abelian subgroup,
- $\Gamma/Z = G$.

Let $g \in G = \Gamma/Z$ and $\rho(g)$ is given representative of coset $g = \rho(g)Z$. Then

$$\rho(g)\rho(h) = \rho(gh)\alpha(g, h),$$

here $\alpha(g, h) \in Z$. Therefore $\alpha : G \times G \to Z$ is mapping ("cocycle" or "system of factors").

If α is given, then multiplying of elements of Γ is also given. Indeed, any element of Γ uniquely presented as $\rho(g)a$, there $g \in \Gamma$, $a \in Z$.

Also,

$$\rho(g)a \cdot \rho(h)b = \rho(g)\,\rho(h)\,\rho(h)^{-1}a\,\rho(h)\,b =$$
$$= \rho(gh) \cdot \alpha(g,h)\,(\rho(h)^{-1}\,a\,\rho(h))\,b.$$

By choosing of representatives $\rho(g)$ we can bound function α depending only on n.

There are only finitely many different functions α if n is fixed. Therefore number of extension Z with G is finite.

6.3 Bound for cocycles

We will use the proof of theorem above. Firstly we will construct such Γ-invariant euclidean metric of $\mathbb{R}^n$ as all nonzero shortest vectors of lattice Z have unit length and contain basis $\mathbb{R}^n$.

Let $a_1, \dots, a_n$ be n linearly independent shortest vectors of Z. Let P be a parallelepiped spanned by $a_1, \dots, a_n$.

Then

$$\mathbb{R}^n = \bigcup_{g \in \Gamma} gP.$$

Since all edges P have unit length, then $\operatorname{diam} P \leqslant n$.

We can choose element $\rho(g)$, which move center of P at minimal distance, as representative of coset g.

Since $g = \rho(g)Z$ and Z-translations of parallelepiped P covers $\mathbb{R}^n$ then this distance is $\leqslant n$.

Therefore $\alpha(g,h) = \rho(gh)^{-1}\rho(g)\rho(h)$ move the center of P at distance $\leqslant 3n$.

Number of such elements of Z is bounded because number $N(r)$ of lattice points Z in ball with radius r and center at origin is at most $(2r+1)^n$ (lemma 1).

Therefore number of functions α bounded depending on $|G| \leqslant f(n)$ and $n = \operatorname{rk} Z$.

The proof of the theorem is complete.

6.4 Remarks

Remark 1 Bieberbach proved more. He proved that abstract isomorphism of crystallographic groups can be realized as affine transformation.

Remark 2 System of factors is not arbitrary function. Since group operation is associative, some condition holds for that function. There is the origins of extension theory for groups with abelian kernel and cohomological theory for groups.

Remark 3 Exact number of isomorphic classes n-dimensional crystallographic groups at $n = 1,\ 2,\ 3,\ 4$ is numbers 2, 17, 219, 4783 correspondingly.

6.5 Pictures

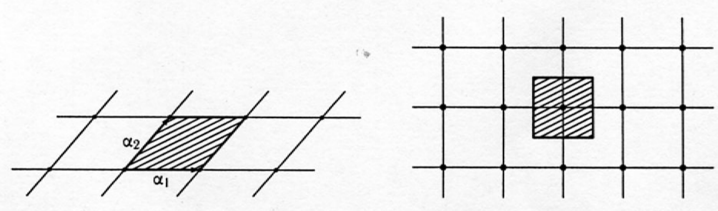
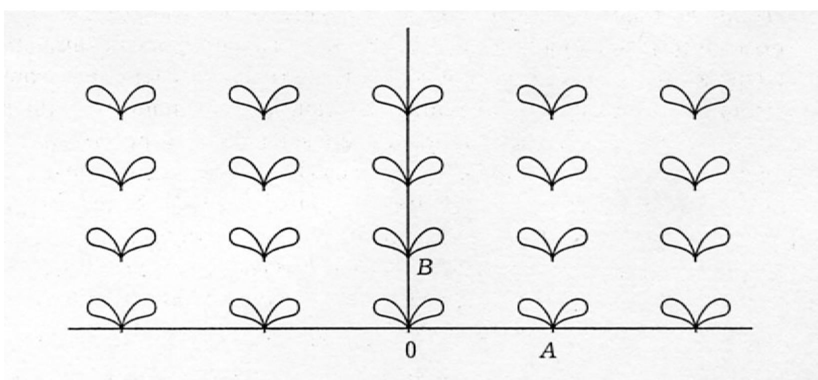
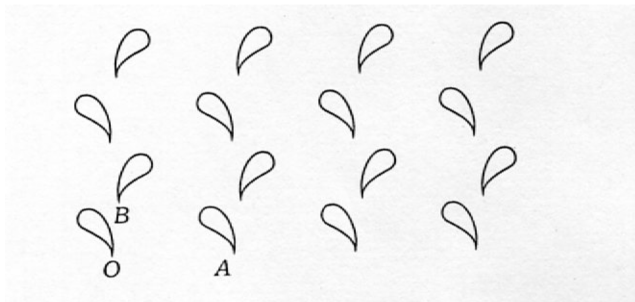


Рис. 6.2. Примитивная ячейка

Рис. 6.3. Ячейка Вигнера—Зейтца





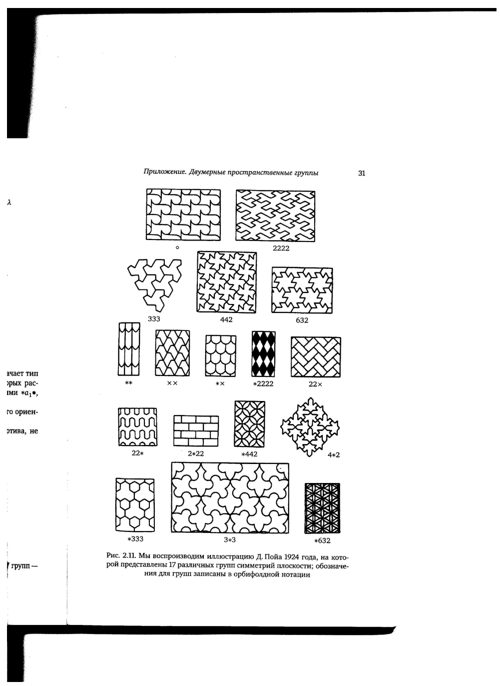


Рис. 2.11. Мы воспроизводим иллюстрацию Д. Пойа 1924 года, на которой представлены 17 различных групп симметрий плоскости; обозначения для групп записаны в орбифолдной нотации

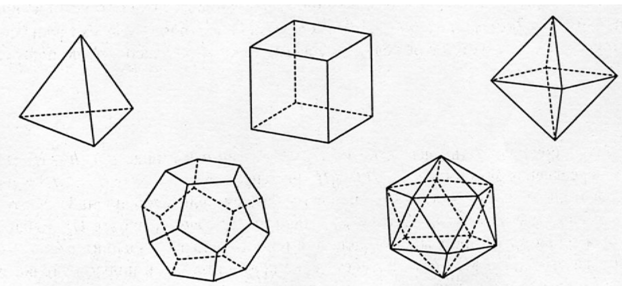
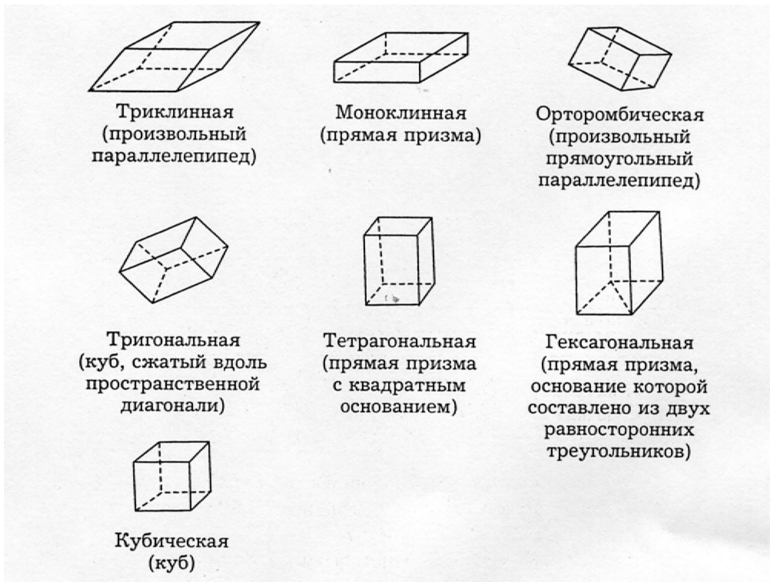


Рис. 6.4. Тетраэдр, куб, октаэдр, додекаэдр, икосаэдр



7 ALGEBRAIC INTEGERS AND ROOTS OF UNITY

Definition. Complex number λ is *algebraic*, if

$$\exists a_1, \ldots, a_n \in \mathbb{Q} : \quad \lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0.$$

Polynomial $x^n + a_1x^{n-1} + \ldots + a_n$ is *annulator* for λ.

If it's degree is minimal, then it is *minimal (annulator)* for λ. It's degree is called *degree of number* λ.

If it is possible to choose $a_1, \ldots, a_n \in \mathbb{Z}$ then λ is called *integer algebraic* number.

Lemma 1) Minimal polynomial divide any annulator polynomial.

2) Minimal polynomial is unique.

3) The set of all algebraic integers is a ring $\overline{\mathbb{Z}}$.

4) Minimal polynomial for algebraic integer has *integer* coefficients.

Lets prove, for example, 4). All roots of annulator polynomial with integer coefficients is algebraic integers.

Using 1) we can see that minimal polynomial for one of them divide annulator.

Therefore all roots of minimal polynomial is integer algebraic numbers.

All coefficients of this polynomial can be expressed through its roots by Viet formulas. Thus, in view of 3), all coefficients are algebraic integers.

Since it is also rational then it is just integer numbers.

Definition Number $\lambda \in \mathbb{C}$ is called *primitive root of unity degree* k if

$$\lambda^k = 1, \ \lambda^l \neq 1 \ \forall l : \ 0 < l < k.$$

Theorem Let $K/\mathbb{Q}$ be algebraic extension degree n. Let $\lambda \in K$ be primitive root degree k. Then

$$k \leqslant 2n^2.$$

Proof. All primitive roots degree k is exactly roots of k-th cyclotomic polynomial $\Phi_k(x), \deg \Phi_k(x) = \varphi(k)$, there φ is phi-function of Euler.

If $k = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, where $2 \leqslant p_1 < p_2 \dots < p_s$ is prime numbers, then

$$\varphi(k) = k \prod_i (1 - \frac{1}{p_i}) =$$

$$= \frac{k}{p_1 \dots p_s} \prod_i (p_i - 1) \geqslant (p_1 - 1) \dots (p_s - 1) \geqslant 2^{s-1}.$$

By the conditions $n \geqslant \varphi(k)$.

Then $2n \geqslant 2\varphi(k) \geqslant 2^s$, $\log_2 2n \geqslant s$.

Therefore

$$n \geqslant \varphi(k) \geqslant k \prod_{i=1}^{s} (1 - \frac{1}{p_i}) \geqslant k(1 - \frac{1}{2})^{\log_2 2n} = \frac{k}{2n}.$$

Hence, $2n^2 \geqslant k$. Proof is complete.

Remark This estimation is raw. Lets calculate exactly, which primitive roots k-th degree of unity is belong to extension $K/\mathbb{Q}$ degree 4. By theorem $k \leqslant 2 \cdot 4^2 = 32$.

Direct solution of inequality $\varphi(k) \leqslant 4$ give values $k = 1, \ 2, \ 3, \ 4, \ 5, \ 6, \ 8, \ 10, \ 12$.

Primitive roots degree $k = 2, \ 3, \ 4, \ 6$ belongs to quadratic extension of field $\mathbb{Q}$. Remain possibilities are $5, \ 8, \ 10, \ 12$.

Rotation symmetries of namely orders $5, \ 8, \ 10, \ 12$ has been experimentally observed of quasicrystalls.

8 ANGLES OF ROTATIONS IN 2-DIM QUASICRYSTALLOGRAPHIC GROUP

8.1 2D general case

Theorem For a real number φ there exists a 2D quasicrystallographic group containing a rotation by φ if and only if

$$\lambda = \exp(i\varphi) = \cos\varphi + i\,\sin\varphi$$

is an algebraic integer.

Proof. Let $\Gamma = \Gamma(G, Z)$ be some quasicrystallographic group of $\mathbb{R}^2$ with quasilattice Z. Let $g \in \Gamma$ be a rotation of $\mathbb{R}^2$ by angle φ.

We can identify the center of the rotation and the origin. We will identify $\mathbb{R}^2$ with the field $\mathbb{C}$. Then $g : z \mapsto \lambda z,\ z \in \mathbb{C}$. Since $|\lambda| = 1$, it follows that $\lambda = \exp(i\varphi)$.

Since $Z \simeq \mathbb{Z}^n$, it follows that $g|_Z \in \operatorname{Aut} Z \simeq GL_n(\mathbb{Z})$.

Every integer unimodular matrix is a root of it's monic characteristic polynomial $p(x)$ with integer coefficients. Thus $p(g|_Z) = 0$, $p(g)z = p(\lambda)z = 0, \forall z \in Z$. Therefore $p(\lambda) = 0$ and λ is an integer algebraic number.

Conversely, let

$$\lambda^n + a_1\lambda^{n-1} + \ldots + a_n = 0, \quad a_i \in \mathbb{Z},\ n = \min.$$

Then $p(x) = x^n + a_1x^{n-1} + \ldots + a_n$ is the minimal polynomial for λ.

Since $\overline{\lambda} = \lambda^{-1}$ and $p(\overline{\lambda}) = 0$, it follows that

$$\frac{1}{\lambda^n} + a_1\frac{1}{\lambda^{n-1}} + \ldots + a_n = 0,$$

$$a_n\lambda^n + a_{n-1}\lambda^{n-1} + \ldots + 1 = 0,$$

$$\lambda^n + \frac{a_{n-1}}{a_n}\lambda^{n-1} + \ldots + \frac{1}{a_n} = 0.$$

The uniqueness of the minimal polynomial yields

$$a_n = \pm 1,\ a_1 = \pm a_{n-1},\ a_2 = \pm a_{n-2}, \ldots.$$

Let's identify euclidean plane $\mathbb{R}^2$ and $\mathbb{C}$. Consider quasilattice

$$Z = \langle 1, \lambda, \lambda^2, \dots, \lambda^{n-1} \rangle.$$

Let $g : z \mapsto \lambda z,\ z \in \mathbb{C}$. Since $|\lambda| = 1$, g is a rotation.

Let $G = \langle g \rangle = \{g^k \,|\, k \in \mathbb{Z}\}$ is a cyclic group and $\Gamma = \Gamma(G, Z)$ is the semidirect product of Z and G.

Then Γ is a quasicrystallographic group, $g \in \Gamma$, g is a rotation of $\mathbb{R}^2$ by angle φ.

8.2 Example

Example Two-dimensional quasicrystallographic group with *infinite cyclic rotation group.*

Let's consider irreducible polynomial of degree 4 with integer coefficients such that two of its roots have absolute value equal to 1 and two others are mutually inverse positive real numbers. For example, let

$$p(x) = (x^2 - (2 - \sqrt{3})x + 1)(x^2 - (2 + \sqrt{3})x + 1) = x^4 - 4x^3 + 3x^2 - 4x + 1.$$

Put $A = 2 - \sqrt{3},\ B = 2 + \sqrt{3}$. Since numbers A and B are irrational, $p(x)$ is irreducible over $\mathbb{Q}$.

It's easy to see that roots $\lambda,\ \overline{\lambda}$ of polynomial $x^2 - Ax + 1$ have absolute value 1 and roots $\mu,\ \mu^{-1}$ of polynomial $x^2 - Bx + 1$ are real and mutually inverse.

Assume that $\mu > 1$. If $\lambda^k = 1$ then since $p(x)$ is the minimal polynomial for λ it divides $x^k - 1$. But then $\mu^k = 1$; a contradiction.

Consequently, $\lambda^k \neq 1$ for all $k \neq 0$. Let's construct the quasicrystallographic group $\Gamma = \Gamma(G, Z)$ using this λ as in the theorem above.

This group is a required example.

Remark 1 All rotations in crystallographic groups have finite orders.

There exist quasicrystallographic groups containing a rotation of infinite order.

Remark 2 Let A be a linear operator of $\mathbb{R}^4$ with unimodular matrix, whose characteristic polynomial is equal to $p(x)$. Then $\mathbb{R}^4$ decomposes into a direct sum of two A-invariant planes.

Furthermore, A preserves an euclidean structure and acts as a rotation in one of these planes, and acts as a hyperbolic rotation in the second plane.

Therefore, we conclude that operator A preserves a pseudo-euclidean structure of type (3,1).

8.3 Rank 3

Corollary There are no two-dimensional quasicrystallographic groups with infinite rotation group and *translation subgroup of rank 3.*

Proof. Assume that there exists a quasicrystallographic group $\Gamma(G, Z) \subset \mathrm{Is}\,\mathbb{C}$ with quasilattice Z of rank 3.

Since $\mathrm{rk}\, Z = 3$, if a rotation of G has finite order then this order is less than $2 \cdot 3^2 = 18$.

The group generated by all rotations of finite orders is a finite cyclic group. Consequently, there exists a rotation $g : z \mapsto \lambda z$ of infinite order with $|\lambda| = 1$.

By the theorem λ is a root of an irreducible over $\mathbb{Q}$ minimal polynomial $p(x)$, $\deg p(x) \leqslant \mathrm{rk}\, Z = 3$.

If $\deg p(x) = 1$ then $p(x) = x \pm 1$, $\lambda = \mp 1$ and $\lambda^2 = 1$.

If $\deg p(x) = 2$ then $p(x) = x^2 - mx + 1$ with $m \in \mathbb{Z}$, $p(\lambda) = p(\overline{\lambda}) = 0$, $\lambda + \overline{\lambda} = 2 \cos \varphi = m$, $|m| \leqslant 2$.

If $m = 0$ then $p(x) = x^2 \pm 1$ and $\lambda^4 = 1$.

If $m = -1$ then $p(x) = x^2 + x + 1$ and $\lambda^3 = 1$.

If $m = 1$ then $p(x) = x^2 - x + 1$ and $\lambda^6 = 1$.

If $m = \pm 2$ then $p(x) = (x \pm 1)^2$ is reducible.

Consequently, $\deg p(x) = 3$ and by the theorem $p(x) = x^3 + mx^2 + mx + 1$ (or $p(x) = x^3 + mx^2 - mx - 1$).

Since $\overline{\lambda} \neq \lambda$ is a root of $p(x)$, applying Viet formula to the last coefficient we infer that the third root of $p(x)$ is equal to ± 1.

This contradicts to irreducibility of $p(x)$. The corollary is proved.

9 ANGLES OF ROTATIONS FOR QUADRATIC QUASILATTICES

9.1 Definition, examples

Definition n-dimensional quasilattice $Z \subset \mathbb{R}^n$ is called *quadratic* if there exists a real quadratic extension K of the field $\mathbb{Q}$ such that

$$Z \subset Ke_1 + \ldots + Ke_n$$

for some $e_1, \ldots, e_n$ in Z.

Example 1 Quasilattice generated in the field of complex numbers by the roots of equation $x^5 = 1$ is quadratic for $K = \mathbb{Q}(\sqrt{5})$.

Indeed, if $\lambda = \cos(2\pi/5) + i\sin(2\pi/5)$ then $\cos(2\pi/5) = (\sqrt{5} - 1)/4$, $\cos(4\pi/5) = 2 - \sqrt{5}$.

Let us denote $L = K \cdot 1 + K \cdot \lambda$. Then L contains

- $i\sin(2\pi/5)$,
- $i\sin(4\pi/5)$,
- λ^2,
- $\lambda^3 = \overline{\lambda^2}$,
- $\lambda^4 = \overline{\lambda}$,
- $Z = \langle 1, \lambda, \lambda^2, \lambda^3, \lambda^4 \rangle$.

Example 2 Quasilattice, generated in $\mathbb{R}^3$ by vectors corresponding to the vertices of the regular icosahedron (dodecahedron), is quadratic.

Indeed, let $\tau: \ \tau^2 = \tau + 1, \ \tau = (\sqrt{5} + 1)/2$.

Vertices of the icosahedron have the following coordinates

$$(0, \ \pm\tau, \ \pm 1), \quad (\pm 1, \ 0, \ \pm\tau), \quad (\pm\tau, \ \pm 1, \ 0).$$

Vertices of the dodecahedron have the coordinates

$$(\pm 1, \ \pm 1, \ \pm 1), \quad (0, \ \pm\tau^{-1}, \ \pm\tau),$$

$$(\tau, \ 0, \ \pm\tau^{-1}), \quad (\pm\tau^{-1}, \ \pm\tau, \ 0).$$

(see, for instance, M. Berger, Geometry, chapter 12, 12.5.5.3, 12.5.5.5).

Remark From the viewpoint of applications in chemistry and physics it is important to classify the angles of rotations in two- and three-dimensional quasicrystallographic groups with quadratic quasilattice. For such groups the rank of the quadratic quasilattice is equal to 4 and 6 correspondingly.

9.2 Lemma, corollary

Lemma Let $\lambda \in \mathbb{C}, \quad |\lambda| = 1$. The number λ is integer algebraic of degree 4 if and only if the number $\overline{\lambda} + \lambda$ is integer algebraic of degree 2.

More exactly,

$$\overline{\lambda} + \lambda = \frac{m \pm \sqrt{m^2 - 4k}}{2},$$

with

$$m, \ k \in \mathbb{Z}, \ |\frac{m \pm \sqrt{m^2 - 4k}}{2}| < 2$$

for some choice of the sign and irrational number $\sqrt{m^2 - 4k}$.

Proof. Let $A = \overline{\lambda} + \lambda \in \mathbb{R}$. As $\overline{\lambda} = \lambda^{-1}$ we have that

$$\lambda^2 - A\lambda + 1 = 0.$$

Minimal polynomials for λ and $\overline{\lambda}$ coincide, so A is an algebraic integer.

If A is rational then A is integer and λ is a number of degree 2, which contradicts our assumption. Therefore, A must be irrational.

Let

$$p(x) = x^4 - mx^3 + nx^2 - ax + b$$

be the minimal polynomial for λ. Then

$$p(x) = (x^2 - Ax + 1)(x^2 - Bx + C).$$

Comparing the corresponding coefficients we obtain the equations

$$m = A + B, \ n = C + 1 + AB, \ a = CA + B, \ b = C.$$

It is obvious that $a - m = CA - A = (C-1)A = (b-1)A$.

If $b \neq 1$ then $A = (a-m)/(b-1) \in \mathbb{Q}$, which contradicts irrationality of A.

Therefore, $b = 1 = C,\ a = m,\ n = 2 + AB$.

Denoting $k = AB = n - 2 \in \mathbb{Z}$ we obtain the equations

$$A + B = m, \quad AB = k.$$

Solving the quadratic equation $x^2 - mx + k = 0$ we get the required presentation:

$$\overline{\lambda} + \lambda = A = \frac{m \pm \sqrt{m^2 - 4k}}{2}, \quad B = \frac{m \mp \sqrt{m^2 - 4k}}{2}.$$

Now the proof of sufficiency is an easy excercise.

Corollary If the point group of a two-dimensional quasicrystallographic group acts irreducible on $\mathbb{Q}$-completed quasilattice, then the quasilattice is of the rank 4 if and only if it is quadratic.

Proof. If $g : z \mapsto \lambda z,\ z \in \mathbb{C}$ is a rotation element in the point group then $|\lambda| = 1$ and λ must be an algebraic integer of degree 4,

otherwise the action of the point group would be reducible because of the commutativity of the group of rotations.

Let us put $A = \overline{\lambda} + \lambda$, $K = \mathbb{Q}(\sqrt{m^2 - 4k})$.

Then $A \in K$ and

$$\lambda^2 - A\lambda + 1 = 0,$$

$$\lambda^2 = A\lambda - 1 \in L = K \cdot 1 + K \cdot \lambda,$$

and $\lambda^3 = A\lambda^2 - \lambda \in L$, $Z \subset L$.

On the contrary, if $Z \subset Ke_1 + Ke_2$ and $|K : \mathbb{Q}| = 2$ then $\operatorname{rk} Z \leqslant 4$.

By the corollary on the quasilattices of rank 3 we have that

$$\operatorname{rk} Z \neq 3$$

and

$$\operatorname{rk} Z \neq 2,$$

otherwise Z would not be a quasilattice.

Therefore, $\operatorname{rk} Z = 4$. The proof of the corollary is complete.

9.3 Theorems 2D and 3D

Theorem An angle φ is the angle of rotation in some 2-dimensional quasicrystallographic group with quadratic quasilattice if and only if

$$\cos\varphi = \frac{m+n\sqrt{d}}{4}, \quad m,\ n \in \mathbb{Z},\ d \in \mathbb{N},\ 4 \,|\, (m^2 - n^2 d).$$

Proof. Because of the lemma we have

$$\cos\varphi = \frac{m \pm \sqrt{m^2 - 4k}}{4}.$$

Let $d = m^2 - 4k$, $\quad n = 1$ if the sign of the square root is 'plus' and $n = -1$, if the sign of the square root is 'minus'. Then $m^2 - n^2 d = 4k$.

Vice versa, if $m^2 - n^2 d = 4k$ then

$$\cos\varphi = \frac{m+n\sqrt{d}}{4} = \frac{m \pm \sqrt{m^2 - 4k}}{4}.$$

The proof is complete.

Theorem An angle φ is the angle of rotation in some 3-dimensional quasicrystallographic group with quadratic quasilattice if and only if

$$\cos\varphi = \frac{m+n\sqrt{d}}{4}, \quad m,\ n \in \mathbb{Z},\ d \in \mathbb{N},\ 4 \,|\, (m^2 - n^2 d).$$

Proof. Sufficiency. Let the condition on the cosine of the angle φ hold. Let us construct a 3-dimensional quasicrystallographic group with the given angle of rotation.

Because of the theorem 1 there exists a 2-dimensional quasicrystallographic group Γ whose point group consists of rotations by the angles which are multiples of $\pm\varphi$.

Its subgroup of translations Z is a quadratic quasilattice: $\mathbb{Q} \otimes Z$ has the dimension 2 over the field $K = \mathbb{Q}(\cos\varphi)$.

Let us include $\mathbb{R}^2$ into $\mathbb{R}^3$ and choose a vector $e \perp \mathbb{R}^2$.

Let $\overline{Z}$ be the quasilattice in $\mathbb{R}^3$ generated by $Z,\ e,\ e\cos\varphi$.

Let $\overline{G}$ consists of rotations around the axis $\mathbb{R}e$ by angles which are multiples of $\pm\varphi$.

Then the subgroup $\overline{\Gamma}$, generated by $\overline{Z}$ and $\overline{G}$ is crystallographic and the quasilattice $\overline{Z}$ is quadratic.

Necessity. Let $\Gamma(G, Z)$ be a 3-dimensional quasicrystallographic group with quadratic quasilattice and let $g \in \Gamma$ be a rotation of $\mathbb{R}^3$ by the angle φ around some axis.

Then its matrix A is defined over the field $K = \mathbb{Q}(\sqrt{d})$.

As $\det(A - I) = 0$, the subspace $W = \mathrm{Im}\,(A - I)$ is defined over the field K and it has the real dimension 2.

Then $T = Z \cap W$ is a quadratic quasilattice in the plane W.

Therefore, the rotation $g \in \Gamma$ induces the rotation by the angle φ in the plane W and it preserves the quadratic quasilattice T. Because of theorem 1 we obtain the formula for $\cos\varphi$.

The proof of the theorem is complete.

10 WILD 4D AND 3D QUASICRYSTALLOGRAPHIC GROUPS

10.1 4D and quaternions

Wild 4D and 3D quasicrystallographic groups

Our goal now is to construct examples of 4-dimensional and 3-dimensional quasicrystallographic groups with a quadratic quasilattice,

whose rotation group is similar in some sense to a *free nonabelian group*.

Let us identify $\mathbb{R}^4$ and the algebra of quaternions $\mathbb{H}$ with orthonormal basis $1,\ i,\ j,\ k$.

Let us put

$$g = \exp(i\varphi), \quad h = \exp(j\psi), \quad G = \langle g,\ h\rangle.$$

Consider the additional requirement

$$0 < \varphi < \pi, \quad |\cos\varphi| < \frac{1}{2},\ \cos\varphi = \frac{m + n\sqrt{d}}{4},$$

$$m,\ n \in \mathbb{Z},\ d \in \mathbb{N},\ 4\,|\,(m^2 - n^2 d);$$

$$0 < \psi < \pi, \quad \cos\psi = 2\cos\varphi.$$

Choose numbers d, m, n in order to make the angles to be irrational: $\varphi,\ \psi \notin \mathbb{Q}\pi$.

It's easy to see that the group G is dense in $SU_2 = S^3 = \{q \in \mathbb{H} \mid |q| = 1\}$.

By Tits theorem the group G contains a free nonabelian subgroup, otherwise it would be virtually solvable,

so it would satisfy the corresponding identity and, therefore, its closure SU_2 would be virtually solvable.

But there is well known fact that SU_2 contains free nonabelian subroups and, consequently, it cannot be a virtually solvable group.

Let now $Z = \mathbb{Z}[G]$ is a $\mathbb{Z}$-module in the additive group of the algebra $\mathbb{H}$, generated by all quaternions contained in G:

$$g^{l_1}h^{l_2}g^{l_3}\dots g^{l_{s-1}}h^{l_s}.$$

Here

$$l_1,\ l_2,\ \dots, l_s \in \mathbb{Z}.$$

Theorem In the previous notations the semidirect product $\Gamma = Z \rtimes G$ is a 4-dimensional quasicrystallographic group with

- the quadratic quasilattice $Z \subset \mathbb{H}$ of rank 8,
- the point group G which is dense in $SU_2 = \{q \in \mathbb{H} : |q| = 1\}$.

Proof. In fact all we need to prove is that Z is a free abelian group of rank 8.

From requirements for $\cos\varphi$ and $\cos\psi$ it follows that g and h are algebraic integers of degree 4 over $\mathbb{Q}$.

Therefore we can assume that $l_i = 0,\ 1,\ 2,\ 3$ for all i.

Let us write the following quaternions in the scalar-vector form

$$g = \alpha + u,\ \alpha = \cos\varphi,\ u = i\sin\varphi,$$

$$h = \beta + v,\ \beta = \cos\psi = 2\cos\varphi, \quad v = j\sin\psi.$$

Then

$$gh = \alpha\beta + \alpha v + \beta u - (u,\, v) + [u,\, v],$$
$$h\overline{g} = \beta\alpha + \alpha v - \beta u + (v,\, u) - [v,\, u],$$
$$gh - h\overline{g} = 2\beta u = i \cdot 4\cos\varphi \cdot \sin\varphi =$$
$$= (g + \overline{g})(g - \overline{g}) = g^2 - g^{-2} \in \mathbb{Z}[g].$$

Therefore,

$$gh = h\overline{g} + z, \quad \overline{g} = g^{-1}, \quad z \in \mathbb{Z}[g].$$

Finally, the subgroup Z is a $\mathbb{Z}$-hull of the set of products

$$h^l g^m, \quad 0 \leqslant l,\ m < 4,$$

and

$$\operatorname{rk} Z \leqslant 16.$$

Let us show now that we can assume that $0 \leqslant l < 2$.

Indeed,

$$h^2 + 1 = 2h \cdot \cos\psi, \quad \cos\psi = 2\cos\varphi = g + g^{-1} = P(g),$$

where $P(x) \in \mathbb{Z}[x]$, $\deg P(x) < 4$.

Then

- $h^2 = -1 + 2h \cdot P(g)$,
- $h^3 = -h + 2h^2 \cdot P(g) = -h + 2(-1 + 2h \cdot P(g)) \cdot P(g)$.

It allows us to eliminate h^2 and h^3 in the products $h^l g^m$.

Consequently, $\operatorname{rk} Z \leqslant 8$.

Prove now that Z is quadratic over $K = \mathbb{Q}(\sqrt{d})$.

As

$$g + g^{-1} = 2\cos\varphi = \frac{m + n\sqrt{d}}{2} = A \in K,$$

we have that

$$g^2 = -1 + Ag,\ g^3 = -g + Ag^2 = -A + (-1 + A^2)g.$$

From this we have

$$Z \subset K1 + Kg + Kh + Kgh.$$

The proof is complete.

10.2 3D example 1

Corollary 1 There exists a 3-dimensional quasicrystallographic group with a quadratic quasilattice of rank 6 and

an infinite nonabelian rotation group which is dense in SO_3.

Proof. Let $\mathbb{R}^3$ be the euclidean space of the pure imaginary quaternions with the basis $i,\ j,\ k$ from the algebra $\mathbb{H}$.

Then the group SU_2 of quaternions of the norm equal 1 acts on $\mathbb{R}^3$ by conjugation

$$\widehat{g} : v \mapsto gvg^{-1},\ v \in \mathbb{R}^3.$$

In the previous example we had $G \subset SU_2$ and $Z = \mathbb{Z}[G]$. It is clear that Z is invariant under conjugation by elements of G. Then $T = Z \cap \mathbb{R}^3$ is also G-invariant.

The group Z is generated as an additive subgroup by the elements

$$1,\ g,\ g^2,\ g^3,\ h,\ hg,\ hg^2,\ hg^3.$$

Then the generators for $T = Z \cap \mathbb{R}^3$ are twice the vector parts of the generators for Z:

$$g - \overline{g} = 2i \sin\varphi, \quad g^2 - \overline{g}^2 = 2i \sin 2\varphi, \ldots .$$

If $g = \alpha + u, \quad h = \beta + v$ then the vector part of gh is of the form $\alpha v + \beta u + [u,\, v]$.

As $\alpha\beta \in K = \mathbb{Q}(\sqrt{d})$ and the coordinates $u,\ v$ in the basis $i,\ j,\ k$ are in K, it follows that $T \subset K\cdot i + K\cdot j + K\cdot k$ and the quasilattice T is quadratic of rank 6.

The group G acts on T by conjugation.

Let H be the image of G in SO_3 under the representation induced by the conjugation action.

Then the semidirect product $\Delta = T \leftthreetimes H$ is a required quasicrystallographic group.

Note that H is dense in SO_3

and by Tits theorem it contains a free nonabelian subgroup F of an infinite rank.The proof of the corollary is complete.

10.3 3D example 2

Corollary 2 There exists a 3-dimensional quasicrystallographic group which is *not contained as a subgroup in a finitely generated* 3-dimensional quasicrystallographic group.

Proof. Let F be a free group of the countable rank from the Corollary 1 with a basis $g_i, \quad i \in \mathbb{N}$. Then F preserves a quadratic quasilattice T.

Let $\{\alpha_i\}_{i \in \mathbb{N}}$ be a countable sequence of real numbers which are algebraically independent over $\mathbb{Q}$.

Choose a vector $v \in \mathbb{R}^3, \quad v \neq 0$.

Let

$$h_i : x \mapsto g_i(x) + \alpha_i v, \quad x \in \mathbb{R}^3.$$

Let $H = \langle h_i \mid i \in \mathbb{N} \rangle$.

As $h_i T h_i^{-1} = T$ then $\Gamma = HT$ is a quasicrystallographic group.

Besides that, Γ is not contained in a finitely generated quasicrystallographic group, otherwise all α_i would belong to a finite extension of the field $\mathbb{Q}$.

The proof of the corollary is complete.

11 PSEUDOORTHOGONALITY OF THE ACTION OF A POINT GROUP ON A QUASILATTICE

11.1 Lemma

Lemma Let $\Gamma(G, Z)$ be a n-dimensional quasicrystallographic group. Let $V \leqslant Z \otimes \mathbb{Q}$ be a minimal subspace invariant under the action of G on Z by conjugation.

Then there exists a symmetric bilinear form on the space $Z \otimes \mathbb{Q}$ which is non-degenerate on the subspace V and invariant under the action of the group G.

Proof. The set $\overline{Z} = Z \otimes \mathbb{Q} \subset \mathbb{R}^n$ is a subspace over the field $\mathbb{Q}$.

By our assumption the group G preserves the scalar product $B(x, y)$ in the euclidean space $\mathbb{R}^n$.

The image of $\overline{Z} \times \overline{Z}$ in $\mathbb{R}$ under $B(x, y)$ has the dimension $\leqslant N^2$ over $\mathbb{Q}$, where $N = \operatorname{rk} Z$.

Let us choose a basis $\beta_1, \dots, \beta_s$.

Then for $x, y \in \overline{Z}$ we have

$$B(x, y) = \sum_i \beta_i B_i(x, y),$$

where $B_i(x, y) \in \mathbb{Q}$.

As $B(x, y)$ is a bilinear symmetric form on $\overline{Z}$, all of the forms $B_i(x, y)$ are also bilinear symmetric.

As $B(x, y)$ is invariant under the action of G, it follows that all of the forms $B_i(x, y)$ are also invariant under this action.

Because of the positive definiteness we have that $B(x, y) \neq 0$ on V.

Therefore, there exist rational numbers α_i (α_i is close to β_i)

such that the form $S(x, y) = \sum \alpha_i B_i(x, y)$ is not identically zero on V.
Besides that, we have that $S(x, y)$ is invariant under the action of G.

Let $\operatorname{Ker} S$ be the kernel of the symmetric bilinear form $S(x, y)$. Then

$$0 \leqslant V \cap \operatorname{Ker} S < V.$$

The subspace $V \cap \operatorname{Ker} S$ is invariant under the action of G.

Therefore $V \cap \operatorname{Ker} S = 0$ because of the minimality of V.

Hence, $S(x, y)$ is non-degenerate on V. The proof of the lemma is complete.

11.2 Theorem

Theorem Let $\Gamma(G, Z)$ be a n-dimensional euclidean quasicrystallographic group.

Then there exists a non-degenerate symmetric bilinear form which is defined on the quasilattice Z,

- taking the integer values on this quasilattice and
- invariant under the action of the point group G on Z by conjugation.

Proof. Let V be a minimal G-invariant subspace in $\overline{Z} = Z \otimes \mathbb{Q}$ and $S(x, y)$ be a G-invariant symmetric bilinear form which is non-degenerate on V.

Let $W = V^{\perp}$ be the orthogonal complement with respect to the form $S(x, y)$. Then $V \cap W = 0$ because of the minimality of V and the fact that $S|_V$ is non-degenerate.

Therefore, $\overline{Z} = V \oplus W$ and W is invariant under G.

Proceeding with this process of decomposition, finally we obtain $\overline{Z} = \oplus_{j=1}^{s} V_j$,

where V_j is a minimal G-invariant subspace, endowed with a non-degenerate G-invariant symmetric bilinear form $S_j(x, y)$, $j = 1, \ldots, s$.

Then the direct sum of these forms gives us the required form on Z.

Multiplying this form by an appropriate natural number we can make it to be integer-valued.

The proof of the theorem is complete.

Remark Garipov R.M. and Churkin V.A. (Siberian Math. J., 2009) proved that in conditions of this theorem every *quasicrystallographic* group $\Gamma = \Gamma(G, Z)$ in euclidean space is isomorphic to a *crystallographic* group in pseudo-euclidean space.

Main problem Found a classification of crystallographic groups on pseudo-Euclidean space of small dimension.

12 MAIN RESULTS

12.1 Introduction

Garipov R. M. (Algebra and Logic, 2003) proved that crystallographic groups on Minkowski spaces uniquely determine their translation lattices as abstract groups. Basing on an analog of the weak Bieberbach theorem, he obtained a full description of some crystallographic classes on Minkowski spaces of small dimension up to isomorphism (Algebra and Logic, 2003; Dokl. Akad. Nauk, 2006; Algebra and Logic, 2008). The weak Bieberbach theorem — ??? He also asked about the validity of the weak Bieberbach theorem for crystallographic groups on arbitrary pseudo-Euclidean spaces.

12.2 Theorems

Here we prove (Churkin V.A., Siberian Mathematical Journal, 2010, Vol. 51, No. 3, P. 700–714)

Theorem 1 If the rotation group of a crystallographic group on a pseudo-Euclidean space includes no normal free abelian subgroup of finite rank greater than 2 which acts identically on a suitable isotropic subspace and on the quotient by it, then the pseudo-Euclidean lattice in this crystallographic group is unique.

Theorem 2 Each crystallographic group on the pseudo-Euclidean space $\mathbb{R}^{p,q}$ with $\min\{p,q\} \leqslant 2$ contains a unique pseudo-Euclidean lattice.

Theorem 3 If $\min\{p,q\} \geqslant 3$ then the pseudo-Euclidean space $\mathbb{R}^{p,q}$ always admits a crystallographic group with at least two distinct pseudo-Euclidean lattices of signature (p,q) and an automorphism exchanging these lattices.

If $\min\{p,q\} = 3$ or $\min\{p,q\} \geqslant 5$ then we can choose these lattices so that the corank of their intersection in each lattice is equal to $\min\{p,q\}$.

If $\min\{p,q\} = 4$ then we can choose these lattices so that the corank of their intersection in each lattice is equal to 3.

Theorem 4 In every crystallographic group with two distinct pseudo-Euclidean lattices of the same rank the corank of their intersection in each lattice must differ from 2 and 4. It can only take the values $3, 5, 6, 7, 8, 9, \dots$.

13 Normal Abelian Subgroups of PEC-Groups

13.1 Definitions

§1. Normal Abelian Subgroups of PEC-Groups

Take a crystallographic group $\Gamma = \Gamma(G, Z)$ in $\mathbb{R}^{p,q}$ with a lattice Z of rank $p+q$ and a rotation group G.

Hence, the translation lattice

(a) is a finite rank normal free abelian subgroup coinciding with its centralizer (a maximal abelian subgroup);

(b) is a subgroup equipped with a nondegenerate real symmetric bilinear form invariant under the conjugation action of the group.

Certainly, abstract isomorphisms preserve these properties. We refer to each subgroup of a crystallographic group with properties (a) and (b) as a *pseudo-Euclidean lattice*. If the form is of signature (p,q) then we also speak of a *lattice of signature* (p,q).

Suppose that two crystallographic groups $\Gamma = \Gamma(G, Z)$ and $\Gamma' = \Gamma'(G', Z')$ are isomorphic and take an abstract isomorphism $\varphi : \Gamma \to \Gamma'$ of groups.

If $\varphi(Z) \neq Z'$ then $Z \neq \varphi^{-1}(Z') = T$. Since the lattice Z' enjoys properties (a) and (b) as a subgroup of Γ', and $\varphi : \Gamma \to \Gamma'$ is a group isomorphism, the subgroup $T = \varphi^{-1}(Z')$ must enjoy properties (a) and (b) as a subgroup of Γ. Consequently, Γ contains the two distinct lattices Z and T.

13.2 Lemma 1 (on invariance of a form)

Lemma 1 (criterion for the invariance of a form) Given an affine crystallographic group Γ with the lattice Z and a bilinear form $\zeta : Z \times Z \to \mathbb{R}$ on Z, take a finite rank normal free abelian subgroup T of Γ with $T \nsubseteq Z$. Then

(1) TZ is a class 2 nilpotent group with center $T \cap Z$ and commutant $[T, Z] \subset T \cap Z$;

(2) the invariance of ζ under the conjugation action of T on Z is equivalent to the fulfilment of two conditions:

(a) the commutant $[T, Z]$ is isotropic with respect to ζ,

(b) the operators $\hat{a} - \hat{1} : x \mapsto axa^{-1}x^{-1},\ x \in Z$ are skew-symmetric with respect to ζ for all $a \in T$.

13.3 Proof

Proof

(1) It is obvious that TZ and $T \cap Z$ are normal subgroups of Γ, and the quotient

$$TZ/(T \cap Z) \simeq T/(T \cap Z) \times Z/(T \cap Z)$$

is an abelian group. Consequently, the commutant $[TZ, TZ] = [T, Z]$ is included into $T \cap Z$. However, $T \cap Z$ is the center of TZ since Z is its own centralizer. Therefore, TZ is a class 2 nilpotent group.

(2) Suppose that ζ is invariant under the conjugation action of T on Z. Denote by $[a, b] = aba^{-1}b^{-1}$ the group commutator of two elements a and b.

(a) Since ζ is bilinear and invariant under T, for all $a \in T$, $b \in Z$, and $c \in T \cap Z$ we have

$$\zeta(aba^{-1}b^{-1},\ c) = \zeta(aba^{-1},\ c) - \zeta(b, c) =$$

$$= \zeta(aba^{-1}, aca^{-1}) - \zeta(b, c) = \zeta(b, c) - \zeta(b, c) = 0.$$

This means that $\zeta([a,b],\ c) = 0$ for all $a \in T$, $b \in Z$, and $c \in T \cap Z$, or, equivalently,

$$\zeta([T,Z],\ T \cap Z) = \zeta(T \cap Z,\ [T,Z]) = 0. \quad (1)$$

Since $[T,Z] \subseteq T \cap Z$, it follows that $\zeta(x,y) = 0$ for all $x, y \in [T,Z]$.

(b) For all $a \in T$ and $x, y \in Z$ the bilinearity of ζ yields

$$\zeta(axa^{-1}, aya^{-1}) = \zeta([a,x]x,\ [a,y]y) = \zeta([a,x],\ [a,y]) +$$

$$+\zeta([a,x],\ y) + \zeta(x,\ [a,y]) + \zeta(x,\ y). \quad (2)$$

If $\zeta(axa^{-1}, aya^{-1}) = \zeta(x,y)$ then taking the isotropy of $[T,Z]$ into account we deduce that

$$\zeta([a,x],\ y) + \zeta(x,\ [a,y]) = 0. \quad (3)$$

This amounts to the skew symmetry of the operator $\hat{a} - \hat{1}$ with respect to ζ since $(\hat{a} - \hat{1})x = [a,x]$.

Suppose now that conditions 2(a) and 2(b) hold for a bilinear form ζ. Then (3) and (2) imply the invariance of ζ under the conjugation action of $a \in T$ on Z.

The proof of Lemma 1 is complete.

13.4 Remarks on Euclead and Minkovsky spaces

Remark 1 Condition 2(a) means that in $\mathbb{R}^n$ there is a subspace Σ isotropic with respect to ζ which includes the ranges of the operators $\hat{a} - \hat{1}$ for all $a \in T$. In particular, the ranks of $\hat{a} - \hat{1}$ are bounded from above by the dimension of Σ. Thus, if the space is Euclidean then $[T,Z] = 1$ and $T = Z$.

Remark 2 In an arbitrary basis the matrix form of condition 2(b) is

$$((A - I)x)^\top Jy + x^\top J(A - I)y = 0,$$

with the matrix A of the operator $\hat{a}$, the identity matrix I, and the matrix J of the form ζ. Since $J^\top = J$, it follows that $(J(A-I))^\top = -J(A-I)$, which expresses the skew symmetry of $J(A-I)$. If the form is nondegenerate then J is a nondegenerate matrix, and the ranks of $J(A-I)$ and $A - I$ coincide. Thus, if this is a Minkowski space then $\mathrm{rk}(A - I) \leqslant 1$ by Remark 1, and then $J(A - I)$ is a skew-symmetric matrix of rank $\leqslant 1$. However, the rank of every skew-symmetric matrix is even; consequently, $J(A - I) = 0$, $A = I$, and $T = Z$.

13.5 Realization

Corollary Take a subspace Σ in $\mathbb{R}^{p+q}$ isotropic with respect to a symmetric bilinear form ζ of signature (p,q). Take a lattice Z in $\mathbb{R}^{p,q}$ such that $N = Z \cap \Sigma$ is a lattice in Σ. Take the group G consisting of the linear operators A which keep Z and N invariant, act identically on the quotients of the chain $0 < N < Z$ of subgroups, and moreover all operators $A - I$ are skew-symmetric with respect to ζ. Then the subgroup $T = GN$ is a finite rank normal free abelian group in the crystallographic group $\Gamma = GZ$, and the group $G \simeq T/T \cap Z$ under multiplication is isomorphic to the discrete group of matrices $\{A - I \mid A \in G\}$ under addition.

13.6 Proof

Given $A \in G$, the hypotheses yield $(A - I)Z \subset N$ and $(A - I)N = 0$. The operators $A, B \in G$ commute when so do $A - I$ and $B - I$. However, $(A - I)(B - I) = 0 = (B - I)(A - I)$. Moreover, the operators in G preserve (centralize) the elements of N. Thus, $T = GN$ is an abelian group. It has no torsion because $A^k = (I + (A - I))^k = I + k(A - I) \neq I$ for $A \neq I$. Since $G \subset \mathrm{Aut}Z$ and the abelian group of integer matrices is finitely generated, it follows that $T = GN$ is a finite rank free abelian group. Its normality in $\Gamma = GZ$ follows from the inclusion $(A - I)Z \subset N$. The group $\Gamma = GZ = GNZ = TZ$ is a class 2 nilpotent by Lemma 1. The invariance of ζ under the operators in G follows from Lemma 1.

The correspondence $A \mapsto A - I$ is an isomorphism between the multiplicative group and the additive group since

$$0 = (A - I)(B - I) = AB - A - B + I,$$

$$AB - I = (A - I) + (B - I).$$

The proof of the corollary is complete.

14 Proof of Theorem 1

14.1 Defining relators for two lattice

§ 2. Crystallographic Groups with a Unique Lattice
Here we prove Theorems 1 and 2.

Take a crystallographic group Γ with two distinct pseudo-Euclidean lattices Z and T and find defining relations for TZ if this group Γ exists. Assume that the maximal dimension of an isotropic subspace is at least 1.

Choose a basis for Z as a free abelian group which agrees with the chain of subgroups

$$Z > T \cap Z \geqslant [T, Z] > 0.$$

Take a basis $b_1, \ldots, b_r$ for Z modulo $T \cap Z$, a basis $c_1, \ldots, c_m$ for $T \cap Z$ modulo the isolator

$$\sqrt{[T, Z]} = \{x \in T \cap Z \mid \exists k : x^k \in [T, Z]\}$$

of $[T, Z]$ in $T \cap Z$, and a basis $e_1, \ldots, e_n$ for $\sqrt{[T, Z]}$. It is clear that $T \cap Z$ is isolated in Z because $[a^k, b] = [a, b]^k$ for class 2 nilpotent groups, and Z is torsion-free. Thus,

$$b_1, \ldots, b_r, \quad c_1, \ldots, c_m, \quad e_1, \ldots, e_n \quad (4)$$

is a basis for Z, as required.

Choose also a basis

$$a_1, \ldots, a_s, \quad c_1, \ldots, c_m, \quad e_1, \ldots, e_n \quad (5)$$

for T which agrees with the chain of subgroups

$$T > T \cap Z \geqslant [T, Z] > 0.$$

Observe that the distance between the lattices is measured by the numbers r and s equal to the coranks of the intersection $T \cap Z$ in Z and T respectively.

Write

$$[a_i, b_j] = \prod_k e_k^{\lambda_{ij}^k}, \quad \lambda_{ij}^k \in \mathbb{Z}. \quad (6)$$

Together with the commutation relations between a_i, b_j, c_k, e_l we can regard these relations as defining relations for TZ.

On the other hand, λ_{ij}^k determine the conjugation action of T on Z. Indeed, for the linear operator $\hat{a} : x \mapsto axa^{-1}$ on Z we have $\hat{a} - \hat{1} : x \mapsto axa^{-1}x^{-1} = [a, x]$ by the multiplicative notation for the operation in Z.

If A_i is the matrix of $\hat{a}_i$ in the basis (4) then by (6) the subdivision of (4) induces the block structure

$$A_i - I = \begin{pmatrix} O & O & O \\ O & O & O \\ L_i & O & O \end{pmatrix}, \quad L_i = \left(\lambda_{ij}^k\right).$$

Here k is the row index and j is the column index in the $n \times r$ matrix L_i.

14.2 The inequalities for Z

The self-centralization of Z is equivalent to the linear independence of the matrices L_i. Indeed, since TZ is class 2 nilpotent, it follows that

$$\left[\prod_i a_i^{\alpha_i}, b_j\right] = \prod_i [a_i, b_j]^{\alpha_i} =$$

$$= \prod_i \left(\prod_k e_k^{\lambda_{ij}^k}\right)^{\alpha_i} = \prod_k e_k^{\sum\limits_i \alpha_i \lambda_{ij}^k} = 1$$

for all j if and only if $\sum_i \alpha_i \lambda_{ij}^k = 0$ for all j and k; i.e., when the matrices L_i are linearly dependent with coefficients α_i.

Take now the Gram matrix J of ζ in the basis (4). By (1), the symmetry of ζ, and the isotropy of $[T, Z]$, it has the block form

$$J = \begin{pmatrix} P & Q & X \\ Q^\top & R & O \\ X^\top & O & O \end{pmatrix}$$

in accordance with the subdivision of (4), where the important block X has size $r \times n$. Observe that the columns of X are linearly independent since J is nondegenerate. In particular,

$$\operatorname{rk}[T, Z] = n \leqslant r = \operatorname{rk}(Z/T \cap Z). \quad (7)$$

By Lemma 1 the invariance of ζ under the conjugation action of T on Z is equivalent to the skew symmetry of $J(A_i - I)$ for all i. However, $J(A_i - I)$ is equal

$$\begin{pmatrix} P & Q & X \\ Q^\top & R & O \\ X^\top & O & O \end{pmatrix} \cdot \begin{pmatrix} O & O & O \\ O & O & O \\ L_i & O & O \end{pmatrix} = \begin{pmatrix} XL_i & O & O \\ O & O & O \\ O & O & O \end{pmatrix}.$$

Thus, the invariance of ζ under the conjugation action of T on Z is equivalent to the skew symmetry of XL_i for all i:

$$(XL_i)^\top = -XL_i, \quad i = 1, \ldots, s. \quad (8)$$

Conversely, the existence of X, of the indicated size $r \times n$ and maximal rank n, satisfying (8) guarantees the existence of a nondegenerate symmetric form ζ on Z, which is invariant by Lemma 1 under the conjugation action of T on Z since we can include this matrix X into a nondegenerate symmetric matrix, for instance into

$$J = \begin{pmatrix} I & O & X \\ O & I & O \\ X^\top & O & O \end{pmatrix}.$$

Indeed, the matrix identity

$$\begin{pmatrix} I & O & O \\ O & I & O \\ -X^\top & O & I \end{pmatrix} \begin{pmatrix} I & O & X \\ O & I & O \\ X^\top & O & O \end{pmatrix} = \begin{pmatrix} I & O & X \\ O & I & O \\ O & O & -X^\top X \end{pmatrix}$$

yields $\det J = \det(-X^\top X) \neq 0$ since $X^\top X$ is the Gram matrix of the independent system of columns of X.

The matrix X has left inverse; thus, the real linear independence of L_i is equivalent to the real linear independence of the skew-symmetric $r \times r$ matrices XL_i. Since the dimension of the space of skew-symmetric matrices is equal to $r(r-1)/2$, we have

$$s \leqslant \frac{r(r-1)}{2}. \quad (9)$$

14.3 The inequalities for T

Suppose now that the lattice T is also equipped with a nondegenerate symmetric bilinear form τ invariant under the conjugation action of Z on T. Then by Lemma 1, the analog of (1), and the isotropy of $[T, Z]$ with respect to τ, the matrix of τ in the basis (5) has the block form

$$J' = \begin{pmatrix} U & V & Y \\ V^\top & W & O \\ Y^\top & O & O \end{pmatrix}$$

in accordance with the subdivision of the basis, where Y is an $s \times n$ matrix.

If $b \in Z$ then $\hat{b} : x \mapsto bxb^{-1}$ is a linear operator on T. Then $\hat{b} - \hat{1} : x \mapsto bxb^{-1}x^{-1} = [b, x]$ by the multiplicative notation for the operation in T. If B_j is the matrix of $\hat{b}_j$ in the basis (5) then by (6) the subdivision of (5) induces the block structure

$$B_j - I = \begin{pmatrix} O & O & O \\ O & O & O \\ -M_j & O & O \end{pmatrix}, \quad M_j = \left(\lambda_{ij}^k\right).$$

Here k is the row index, and i is the column index in the matrix M_j.

By analogy with the above, the invariance of τ under the conjugation action of Z on T is equivalent to the skew symmetry condition

$$(YM_j)^\top = -YM_j, \quad j = 1, \ldots, r. \quad (10)$$

The columns of Y are linearly independent since J' is a nondegenerate matrix. Thus,

$$n \leqslant s, \quad (11)$$

and Y has left inverse. Consequently, the linear independence of M_j is equivalent to the linear independence of the skew-symmetric $s \times s$ matrices YM_j. The dimension of the space of skew-symmetric matrices is equal to $s(s-1)/2$; thus,

$$r \leqslant \frac{s(s-1)}{2}. \quad (12)$$

14.4 The end of proof theorem 1

It follows from (9) and (12) that the coranks s and r of $T \cap Z$ in T and Z are at least 3. Indeed, if $s \leqslant 2$ then $r \leqslant 1$ by (12), and then $s = 0$ by (9), and $T = Z$. This means that if a crystallographic group in $\mathbb{R}^{p,q}$ includes two lattices, then its rotation group must contain an abelian normal subgroup $T/(T \cap Z)$ of rank $\geqslant 3$ acting identically on the quotients of the flag of subspaces

$$\mathbb{R}^{p,q} > \Sigma \geqslant O,$$

where Σ is the real linear span of the commutant $[T,\ Z]$. The proof of Theorem 1 is complete.

14.5 Lemma 2 (a matrix criterion for two lattices)

Lemma 1 and its translation into the matrix language in the proof of Theorem 1 imply Lemma 2 (criterion for the existence of two lattices) The existence of two pseudo-Euclidean lattices Z and T in a crystallographic group of the form $\Gamma = TZ$ with the coranks r and s of $T \cap Z$ in Z and T respectively, and rank n of $[T, Z]$, is equivalent to the existence of a "cube" $\left(\lambda_{ij}^k\right)$ of integers, where $1 \leqslant i \leqslant s$, $1 \leqslant j \leqslant r$, $1 \leqslant k \leqslant n$, as well as a pair of matrices X and Y of size $r \times n$ and $s \times n$ respectively and the maximal rank n satisfying

(a) all face sections L_i of the cube $\left(\lambda_{ij}^k\right)$ determined by $i = \text{const}$ are linearly independent, and all lateral sections M_j of the cube $\left(\lambda_{ij}^k\right)$ determined by $j = \text{const}$ are linearly independent;

(b) all matrices XL_i and YM_j are skew-symmetric.

15 Proof of Theorem 2

Take a crystallographic group $\Gamma = \Gamma(G, Z)$ in $\mathbb{R}^{p,q}$ with a lattice Z of rank $p + q$ and a rotation group G. Take a lattice T in G with $T \neq Z$.

Firstly examine the case $\min\{p, q\} \leqslant 1$.

For Euclidean spaces condition 2 in Lemma 1 means that T and Z commute. Since Z is a maximal abelian subgroup, this is impossible for $T \neq Z$; i.e., the lattice is unique.

For Minkowski spaces Lemma 1 implies that the operators $\hat{a} - \hat{1}$ are of rank at most 1 and are skew-symmetric with respect to ζ. Since the rank of every skew-symmetric matrix is even, all these operators are zero by Remark 3 following Lemma 1. This means that T and Z commute, and $T = Z$.

Suppose now that the maximal dimension of an isotropic subspace with respect to ζ is equal to 2. We may assume that in a suitable basis the quadratic form corresponding to ζ is

$$-x_1^2 - x_2^2 + x_3^2 + x_4^2 + \cdots + x_n^2.$$

Fix in $\mathbb{R}^{2,q}$, with $q \geqslant 2$, a maximal isotropic subspace

$$\Sigma = \{(\alpha, \beta, \alpha, \beta, 0, \ldots, 0)^\top \mid \alpha, \beta \in \mathbb{R}\}.$$

By Witt's theorem, two arbitrary maximal isotropic subspaces are isometric. Thus, there exists a transformation h of $\mathbb{R}^{2,q}$ preserving ζ and satisfying $h([T, Z]) \subset \Sigma$.

Since

$$\begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} G & Z \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} hGh^{-1} & hZ \\ 0 & 1 \end{pmatrix};$$

we can, if need be, replace $\Gamma = \Gamma(G, Z)$ by a conjugate subgroup in the isometry group of the pseudo-Euclidean space, and assume that $[T, Z] \subset \Sigma$.

If A is the matrix of $\hat{a}$ and J is the matrix of ζ in the standard basis then we find respectively that the columns of $A - I$ are contained in Σ, and $J(A - I)$ must be an ordinary skew-symmetric matrix. However,

$$J(A-I) = \begin{pmatrix} -\alpha & -\gamma & -\lambda & -\nu & \dots \\ -\beta & -\delta & -\mu & -\xi & \dots \\ \alpha & \gamma & \lambda & \nu & \dots \\ \beta & \delta & \mu & \xi & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{pmatrix}.$$

If this matrix is skew-symmetric then $\alpha = \delta = \lambda = \xi = 0$ and $\beta = -\gamma = -\mu = \nu$.

Consequently, the set of matrices

$$J(A-I) = \beta \begin{pmatrix} 0 & -1 & 0 & 1 & 0 & \dots \\ 1 & 0 & -1 & 0 & 0 & \dots \\ 0 & -1 & 0 & 1 & 0 & \dots \\ 1 & 0 & -1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

and then $A - I$ constitutes a one-parameter family.

By the corollary to Lemma 1 it is discrete, and as an additive group it is isomorphic to the image of T acting on Z by conjugation.

Consequently, this family is additively generated by one matrix. Since the kernel of the homomorphism $a \mapsto \hat{a}$, for $a \in T$, coincides with $T \cap Z$ by the self-centralization of Z, the group $T/(T \cap Z)$ is cyclic, and $s = 1$. From (12) it follows that $Z = T$, which contradicts our assumption. The proof of Theorem 2 is complete.

16 Proof of Theorem 3

16.1 A rotation symmetry cube for $p = q$

§ 3. Crystallographic Groups with Two Lattices

Here we prove Theorems 3 and 4.

Use Lemma 2. Construct the required matrices of the forms and a cube of numbers $\left(\lambda_{ij}^k\right)$ in the case $p=q=r=s=n\geqslant 3$, $p\neq 4$, $m=0$. In this case in the matrices J and J' the submatrices Q, R, V, and W are absent. Put $P=U=O$ and $X=Y=I$. Then the bilinear forms ζ and τ are symmetric and of type (p,p). In this case in the $p\times p\times p$ cube $\left(\lambda_{ij}^k\right)$ (for brevity simply a size p cube) all face and lateral sections are skew-symmetric. The double skew symmetry

$$\lambda_{ij}^k=-\lambda_{ik}^j, \quad \lambda_{ij}^k=-\lambda_{kj}^i \quad \forall i,j,k$$

implies the symmetry of the cube with respect to the rotation $i\mapsto j\mapsto k\mapsto i$ about the axis $i=j=k$ through $2\pi/3$:

$$\lambda_{ij}^k=-\lambda_{ik}^j=\lambda_{jk}^i,$$

as well as the skew symmetry of the horizontal sections $k=\mathrm{const}$.

Conversely, the skew symmetry of the face sections only and the symmetry of the cube with respect to the reverse rotation $k\mapsto j\mapsto i\mapsto k$ (equivalently, the rotation $i\mapsto j\mapsto k\mapsto i$) imply the skew symmetry of the other sections:

$$\lambda_{ij}^k=-\lambda_{ik}^j=-\lambda_{kj}^i, \quad \lambda_{ij}^k=-\lambda_{ik}^j=-\lambda_{ji}^k.$$

Now express a cube of size p as the union of embedded cubes of increasing size with the common diagonal $i=j=k$ corresponding to the increasing chain

$$\{p\}\subset\{p,p-1\}\subset\{p,p-1,p-2\}\subset\cdots\subset\{p,p-1,p-2,\dots,1\}$$

of indexing sets. Each cube results from the previous one by adding three outer matrix sections of size increasing by 1. Construct them so that these outer matrix sections result from one by the rotation about the axis $i=j=k$ through $2\pi/3$.

Start with the zero cube of size 2. Then, beginning with size 3, take each of the subsequent outer matrix sections $i=\mathrm{const}$ so that it is an integer skew-symmetric matrix with the zero first row and column. Require in addition that this matrix be of maximal rank and independent of the previous parallel sections. For instance, for $p=3$ the outer sections of the cube may have the form

$$
\begin{array}{ccccccc}
 & & & 0 & & & \\
i\nwarrow & & -1 & & 1 & & \nearrow j \\
 & 0 & & 0 & & 0 & \\
 & & 0 & & 0 & & \\
 & 1 & & 0 & & -1 & \\
 & & 0 & & 0 & & \\
 & 0 & & 0 & & 0 & \\
 & & -1 & & 1 & & \\
 & & & 0 & & & \\
 & & & & & & \\
 & & & k\downarrow & & &
\end{array}
$$

Its face sections

$$
L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}
$$

are linearly independent. Since $M_1 = -L_1$, $M_2 = -L_2$, and $M_3 = -L_3$, the lateral sections are also independent.

16.2 A cube for $p = q$

In the general case the required cube exists for $p > 2$, $p \neq 4$. Let us prove this claim.

If $p \geqslant 3$ is odd then the submatrix of the outer face matrix section resulting by removing the first row and column has an even size $p - 1 \geqslant 2$.

Choose it so that the determinant of the submatrix be nonzero, and columns (and rows) be independent. Under the rotation through $2\pi/3$ the rows of the outer face matrix become the first column s of the lateral matrix sections.

Thus, the lateral sections, with the exception of the outer one, are linearly independent.

If all lateral sections are dependent then the outer section can be linearly expressed in terms of the others. However, its first row is zero; thus, all coefficients of the linear combination are zero, and the outer lateral section is zero. We arrive at a contradiction since we choose all outer sections to be nonzero.

If $p \geqslant 4$ is even then, removing the face, lateral, and upper outer sections, we obtain a cube of odd size $p-1 \geqslant 3$.

Its sections appear as submatrices in the sections of the original size p cube, with the exception of the outer sections. By the previous argument we can choose them linearly independent.

The dimension $(p-1)(p-2)/2$ of the space of skew-symmetric matrices is greater than $p-1$ for $p>4$. Choose the outer section of the size p cube so that its submatrix obtained by removing the first row and column is skew-symmetric and independent of the corresponding submatrices of the other parallel sections. Then all parallel sections are independent.

Observe that for $p=4$ a cube with independent sections cannot exist. In Theorem 4 we prove an even stronger claim.

Let us find an automorphism exchanging the lattices. It is easy to see that the correspondence

$$a_i \longleftrightarrow b_i, \quad c_k \longleftrightarrow c_k, \quad i,k=1,\dots,p,$$

preserves the defining relations of $\Gamma = TZ$, and thus, it extends to an automorphism of Γ exchanging the subgroups T and Z. Indeed, (6) is carried into the valid equality

$$[b_i, a_j] = \prod_k e_k^{\lambda_{ij}^k}$$

since $\lambda_{ij}^k = -\lambda_{ji}^k$ by the skew symmetry of the k-sections in the constructed cube $\left(\lambda_{ij}^k\right)$.

16.3 A cube for $p<q$

By construction, the coranks of the intersection $Z \cap T$ in the lattices Z and T are equal to p.

Suppose now that $p<q$ and $m=q-p$. Decompose the pseudo-Euclidean space $\mathbb{R}^{p,q}$ as the orthogonal sum of pseudo-Euclidean spaces

$$\mathbb{R}^{p,q} = \mathbb{R}^{p,p} \oplus \mathbb{R}^m,$$

where $\mathbb{R}^m$ is a Euclidean space. For $p \geqslant 3$, $p \neq 4$, choose in $\mathbb{R}^{p,p}$ one of the constructed crystallographic groups Γ with two lattices Z and T of type

(p, p). Take an abelian lattice R on the Euclidean space $\mathbb{R}^m$ and consider the direct product $\Gamma' = \Gamma \times R$ of groups. Then Γ' is a crystallographic group with two distinct lattices $Z \times R$ and $T \times R$, which, obviously, we may assume equipped with Γ'-invariant symmetric bilinear forms of type (p, q). Extend an automorphism of Γ exchanging Z and T by the identity on R. This yields an automorphism of the crystallographic group Γ' exchanging $Z \times R$ and $T \times R$. The condition on the coranks of the intersection of the lattices is obviously fulfilled.

It remains to examine the case $p = 4$ and $q \geqslant p$. Apply the previous decomposition. Represent the pseudo-Euclidean space $\mathbb{R}^{4,q}$ as the orthogonal sum of pseudo-Euclidean spaces

$$\mathbb{R}^{4,q} = \mathbb{R}^{3,3} \oplus \mathbb{R}^{1,q-3}.$$

Choose in $\mathbb{R}^{3,3}$ one of the constructed crystallographic groups Γ with two lattices Z and T of signature $(3, 3)$. Take a lattice R in $\mathbb{R}^{1,q-3}$ and consider the direct product $\Gamma' = \Gamma \times R$ of groups. Extend an automorphism of Γ exchanging Z and T by the identity on R. This yields an automorphism of the crystallographic group Γ' exchanging $Z \times R$ and $T \times R$. In this case the coranks of the intersection of the lattices are equal to 3.

The proof of Theorem 3 is complete.

17 Proof of Theorem 4

Proof of Theorem 4

By Theorem 3 together with (9) and (12) it suffices to show that the coranks of the intersection of distinct lattices cannot be simultaneously equal to 4. Use Lemma 2, keep the previous notation, and assume on the contrary that $r = s = 4$.

Split the subsequent argument into steps.

Step 1. The matrices XL_i are linearly independent since L_i are linearly independent and X has left inverse. Similarly, the matrices YM_j are linearly independent as well.

Step 2. All matrices XL_i, YM_j, L_i, and M_j are of rank 2. Indeed, multiplying every column of the cube $\left(\lambda_{ij}^k\right)$ by Y we obtain a new cube with the face sections YL_i and skew-symmetric lateral sections YM_j. Then every face section YL_i contains a zero row and $\operatorname{rk} YL_i < 4$. However, the matrices X and Y have left inverses; thus, $\operatorname{rk} XL_i = \operatorname{rk} L_i = \operatorname{rk} YL_i < 4$. On the other hand, XL_i is skew-symmetric, and its rank is even. Taking the linear independence of XL_i into account, we deduce that $\operatorname{rk} XL_i = 2$. Similarly we can prove that $\operatorname{rk} YM_j = 2$.

Step 3. Elimination of Y.

The matrix Y has the left inverse Y^{-1} of size $n \times 4$, while $Y^{-1}Y = I$ is an $n \times n$ matrix. Then

$$XL_i = X \cdot I \cdot L_i = XY^{-1}YL_i = X'L_i',$$

where $X' = XY^{-1}$, and $L_i' = YL_i$ is a face section of the new cube $\left(\nu_{ij}^k\right)$ resulting from the multiplication of all columns of the old cube by Y. Its lateral sections $M_j' = YM_j$ are skew-symmetric matrices.

Replacing the old cube with the new cube, we can eliminate Y and assume that there exist a cube (now not necessarily of integers) and a 4×4 matrix X of rank n such that all lateral sections M_j of the cube are skew-symmetric, as well as all matrices XL_i for the face sections L_i. Moreover, the matrices M_j are linearly independent and of rank 2. A similar claim holds for XL_i, as well as L_i.

Step 4. If the set of all rows of the concatenation of the matrices L_i, $i = 1, \dots, 4$, is included into a proper subspace of $\mathbb{R}^4$ then all rows satisfy a nontrivial linear equation $\sum_{j=1}^4 \lambda_j x_j = 0$. Then $\sum_{j=1}^4 \lambda_j M_j = 0$, which contradicts the linear independence of the lateral sections M_j. Consequently, the system of rows of the union of L_i includes four independent rows. Denote them by a, b, c, and d. We may assume that they lie below the main diagonals of the skew-symmetric matrices M_j. Denote two other remaining rows by x and y.

Step 5. List the possible locations of the intersections of a, b, c, and d with the skew-symmetric matrices M_j, considering that $\operatorname{rk} L_i = 2$:

$$\begin{pmatrix} 0 & -x_j & -a_j & -b_j \\ x_j & 0 & -c_j & -d_j \\ a_j & c_j & 0 & -y_j \\ b_j & d_j & y_j & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -a_j & -x_j & -b_j \\ a_j & 0 & -c_j & -y_j \\ x_j & c_j & 0 & -d_j \\ b_j & y_j & d_j & 0 \end{pmatrix},$$

$$\begin{pmatrix} 0 & -a_j & -b_j & -x_j \\ a_j & 0 & -y_j & -c_j \\ b_j & y_j & 0 & -d_j \\ x_j & c_j & d_j & 0 \end{pmatrix}.$$

It is easy to verify that there are no other intersections.

Step 6. Verify that the rows x and y of L_i are zero. Indeed, considering that $\operatorname{rk} L_i = 2$, express them in terms of the independent a, b, c, or d in L_i using the occurrences of x, $-x$, y, and $-y$. We have

$$x = \alpha a + \beta b = \gamma c + \delta d, \quad y = \alpha' a + \beta' c = \gamma' b + \delta' d.$$

The linear independence of a, b, c, and d yields $x = y = 0$. Consequently, all sections M_j are either of type I, or of type II, or of type III, where

$$\text{I} : \begin{pmatrix} 0 & 0 & -a_j & -b_j \\ 0 & 0 & -c_j & -d_j \\ a_j & c_j & 0 & 0 \\ b_j & d_j & 0 & 0 \end{pmatrix}, \quad \text{II} : \begin{pmatrix} 0 & -a_j & 0 & -b_j \\ a_j & 0 & -c_j & 0 \\ 0 & c_j & 0 & -d_j \\ b_j & 0 & d_j & 0 \end{pmatrix},$$

$$\text{III} : \begin{pmatrix} 0 & -a_j & -b_j & 0 \\ a_j & 0 & 0 & -c_j \\ b_j & 0 & 0 & -d_j \\ 0 & c_j & d_j & 0 \end{pmatrix}$$

Step 7. In a matrix M_j of either of these types the determinant of the submatrices of the crossings with the rows a, b, $\pm c$, $\pm d$ of size 2 is equal to zero, for otherwise $\det M_j \neq 0$ by Laplace's formula for expanding the determinant by two rows. Consequently, the columns of these submatrices are linearly dependent, and then the corresponding columns of M_j are proportional.

Step 8. Suppose that Xu and Xv are linearly independent columns with indices j and l of the skew-symmetric matrix XL_i of rank 2. Then the submatrix of XL_i with rows and columns j and l cannot be zero, for otherwise XL_i would be a skew-symmetric matrix of rank 2, and arguments similar to step 7 imply that Xu and Xv would be linearly dependent. Hence, the entries (j, l) and (l, j) of XL_i contain opposite *nonzero* numbers; denote them by α and $-\alpha$.

It is clear that u and v constitute a basis for the system of columns with indices j and l of the matrices L_i. Then M_j and M_l by step 7 contain proportional columns λu and μv, which by step 6 lie in the common section L_m, where $m \neq i$. Thus, the skew-symmetric matrix XL_m includes the columns $X(\lambda u) = \lambda(Xu)$ and $X(\mu v) = \mu(Xv)$ with indices j and l, which in the entries (j, l) and (l, j) contain $\lambda\alpha$ and $-\mu\alpha$. By the skew symmetry $\lambda\alpha = \mu\alpha$, and so $\lambda = \mu$. Consequently, the basis columns Xu and Xv for

XL_i, as well as λXu and λXv for XL_m with the same indices j and l, are proportional with the *common* coefficient λ.

Step 9. Verify that $XL_m = \lambda XL_i$, invoking the skew symmetry of the matrices and the condition on the rank. This will yield the required contradiction with the claim of their independence in step 1.

Using arguments similar to steps 4–6, we may assume that all skew-symmetric matrices XL_i are either of type I, or of type II, or of type III. Suppose that it is type I, and

$$XL_i = \begin{pmatrix} 0 & 0 & -\alpha & -\beta \\ 0 & 0 & -\gamma & -\delta \\ \alpha & \gamma & 0 & 0 \\ \beta & \delta & 0 & 0 \end{pmatrix}.$$

Pick columns 1 and 3 as a basis. If $\alpha = 0$ then $\beta \neq 0$ and $\gamma \neq 0$ by the independence of columns 1 and 3. However, then the rank of XL_i is equal to 4 rather than 2. Hence, $\alpha \neq 0$, and δ is uniquely determined in terms of α, β, and γ from the equality $\alpha\delta - \beta\gamma = 0$, which follows from the condition on the rank of XL_i. Thus, XL_i is uniquely reconstructed from columns 1 and 3 using the following conditions: the skew symmetry, type I, and the rank being equal to 2. Multiplying columns 1 and 3 by λ, we obtain the corresponding columns of XL_m. Since XL_m is skew-symmetric, of type I, and of rank 2, it is completely reconstructed from columns 1 and 3, and then $XL_m = \lambda XL_i$.

Similar arguments are valid if we replace the indices of the basis columns or the type.

The proof of Theorem 4 is complete.

18 Example with exactly two lattices

Example with exactly two lattices

Let Γ be the group in $\mathbb{R}^{3,3}$ with cube of numbers (λ_{ij}^k) constructed in proof of theorem 3. Then $\Gamma = TZ$ is nilpotent group of class 2, Z is normal free abelian subgroup with basis $b_1,\ b_2,\ b_3,\ c_1,\ c_2,\ c_3$, T is normal free abelian subgroup with basis $a_1,\ a_2,\ a_3,\ c_1,\ c_2,\ c_3$ and

$$\begin{array}{lll} [a_1, b_1] = 1, & [a_2, b_1] = c_3^{-1}, & [a_3, b_1] = c_2, \\ [a_1, b_2] = c_3, & [a_2, b_2] = 1, & [a_3, b_2] = c_1^{-1}, \\ [a_1, b_3] = c_2^{-1}, & [a_2, b_3] = c_1, & [a_3, b_3] = 1. \end{array}$$

Briefly,

$$[a_i,\ b_i] = 1, \quad [a_i,\ b_j] = c_k^{\varepsilon(i\,j\,k)} \text{ if } i \neq j.$$

Here $[a,\ b] = aba^{-1}b^{-1}$ is the commutator of elements a and b, $\varepsilon(i\,j\,k)$ is the signum of permutation $(i\,j\,k)$.

The group Γ is crystallographic in $\mathbb{R}^{3,3}$ if

$$b_1,\ b_2,\ b_3,\ c_1,\ c_2,\ c_3$$

is a basis of lattice $Z = \mathbb{Z}^6$ of translations of $\mathbb{R}^6$,

the generators a_1, a_2, a_3 induce the linear transformations

$$\widehat{a_i} : x \mapsto a_i x a_i^{-1}, \quad x \in Z \simeq \mathbb{Z}^6$$

with block-matrices

$$A_i = \begin{pmatrix} I & O \\ L_i & I \end{pmatrix}, \quad i = 1, 2, 3,$$

in the basis $b_1,\ b_2,\ b_3,\ c_1,\ c_2,\ c_3$. Here I is identity (3×3)-matrix and

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},\ L_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$L_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

The nondegenerate symmetric bilinear form

$$\zeta(x, y) = x_1 y_4 + x_4 y_1 + x_2 y_5 + x_5 y_2 + x_3 y_6 + x_6 y_3$$

on $\mathbb{R}^6$ has type $(3, 3)$ and is invariant under the action of operators $\widehat{a_i}$.

For example, the associated quadratic form $\kappa(x) = \zeta(x, x)$ is A_1-invariant:

$$\kappa(A_1 x) = \kappa(x_1, x_2, x_3, x_4, -x_3 + x_5, x_2 + x_6) =$$

$$= 2x_1 x_4 + 2x_2(-x_3 + x_5) + 2x_3(x_2 + x_6) = 2x_1 x_4 + 2x_2 x_5 + 2x_3 x_6 = \kappa(x).$$

The automorphism $a_i \leftrightarrow b_i,\ c_i \leftrightarrow c_i$, $i = 1, 2, 3$, permutes the subgroups T and Z.

Therefore T has a Z-invariant nondegenerate symmetric bilinear form.

Theorem (Ch., 2009)

1) The crystallographic group Γ, constructed above, is unique up to isomorphizm under the conditions

a) Γ has two distinct pseudo-Euclidean lattices of rank $\leqslant 6$,

b) Γ is minimal: the derived subgroup and the center of Γ coincide.

2) Γ has *exactly* two pseudo-Euclidean lattices.

3) The group Γ^n is crystallographic and has *exactly* 2^n pseudo-Euclidean lattices.

19 Problems

Problem 1 Conjecture: Every crystallographic group on pseudo-Euclidean space has a *finite number* of pseudo-Euclidean lattices.

Problem 2 Obtain a classification of all crystallographic groups on pseudo-Euclidean space $\mathbb{R}^{2,2}$ for a fixed classic discrete group G of rotations.

Problem 3: Auslender's conjecture, 1964 Properly discontinuous cocompact group of affine transformations of $\mathbb{R}^n$ is virtually solvable.

A survey and new results: Abels H., Margulis G.A., Soifer G.A. *Properly discontinuous groups of affine transformations with orthogonal linear part*, C. R. Acad. Sci. Paris, 1997, T. 324 (Ser.I), P. 253–258.

20 Survey by Moodi R.V.

20.1 Properties of a quasicrystall

Robert V. Moody, Model set: A Survey

This article surveys the mathematics of the cut and project method as applied to point sets Λ, called here model sets. It covers the geometric, arithmetic, and analytical sides of this theory as well as diffraction and the connection with dynamical systems.

Following properties may be considered as representative:

- discreteness
- extensiveness
- finiteness of local complexity

- repetitivity
- diffractivity
- aperiodicity
- existence of exotic symmetry.

1) *Discreteness and extensiveness* of Λ means that Λ is *Delone set.*

2) *Finiteness of local complexity* of Λ means that for each $r > 0$ there are, *up to translation*, only finitely many point sets (called *patches of radius* r) of the form $\Lambda \cap B_r(v)$. Here $B_r(v)$ is the ball of radius r about the point $v \in \mathbb{R}^d$. So, on each scale, there are only finitely many different patterns of points. This condition can be expressed topologically: Λ has finite local complexity iff the closure of $\Lambda - \Lambda$ is discrete.

3) *Repetitivity* of Λ means loosely that any finite patch that appears, appears infinitely often. More precisely, given any patch of radius r there is an R so that within each ball of radius R, no matter its position in $\mathbb{R}^d$, there is at least one translate of this patch.

4) *Diffractivity* of Λ means that the Fourier transform of the autocorrelation density that arises by placing a delta peak on each point of Λ, should contain a part that looks discrete and point-like.

5) *Aperiodicity* of Λ means that Λ has not nontrivial translation symmetry.

6) *Existence of exotic symmetry*: for example, Penrose tiling has a self-similarity with factor τ^4, $\tau = (1+\sqrt{5})/2$, Golden ratio.

20.2 Definition model set

By definition, a cut and project scheme or model set consists of a collection of spaces and mappings

$$\begin{array}{c} \mathbb{R}^n \overset{\pi_1}{\longleftarrow} \mathbb{R}^n \times G \overset{\pi_2}{\longrightarrow} G \\ \cup \\ \tilde{L} \end{array}$$

where

- $\mathbb{R}^n$ is 'physical' space — a real euclidean space

- G is 'internal' space — a locally compact abelian group, for example, $\mathbb{R}^d$
- $\tilde{L}$ is a lattice (discrete and cocompact subgroup) in 'hyperspace' $\mathbb{R}^n \times G$
- π_1 is a projection map and $\pi_1 \mid_{\tilde{L}}$ is injective
- π_2 is a projection map and $\pi_2(\tilde{L})$ is dense in G

Let $L = \pi_1(\tilde{L}) \subset \mathbb{R}^n$ and the map $^* : L \to G$ be defined by the rule

$$x^* = \pi_2(\pi_1^{-1}(x)), \quad x \in L.$$

Let $W \subset G$ be a 'window' with properties

- W is nonempty and $W = \overline{\mathrm{int}(W)}$ is compact
- $\partial W \cap \pi_2(\tilde{L}) = \emptyset$
- ∂W is of Haar measure 0.

Then set

$$\Lambda(W) = \{u \in L \mid u^* \in W\}$$

or

$$Q(W) = \{v \in \tilde{L} \mid \pi_2(v) \in W\}$$

is a *'quasicrystal'*.

Definition (Artamonov, Sanches, 2008-10) Let $G = \mathbb{R}^d$. The group of all affine transformations g of hyperspace $\mathbb{R}^n \times \mathbb{R}^d$ with $g(Q(W)) = Q(W)$ is called a *proper group of symmetries of quasicrystal* $Q(W)$ or $\Lambda(W)$ and is denoted by $\mathrm{Sym}\,(\Lambda(W))$.

The group of all affine transformations g of hyperspace $\mathbb{R}^n \times \mathbb{R}^d$ with

$$g(\tilde{L}) = \tilde{L},\ g(\mathbb{R}^n) = \mathbb{R}^n,\ g(\mathbb{R}^d) = \mathbb{R}^d$$

is called a *group of symmetries* of a quasicrystal and is denoted by $\mathrm{Sym}\,(L)$.

Our euclidean quasicrystallographic groups are subgroups of the group symmetries of a quasicrystal.

20.3 Icosian model set

Icosian model set (Kramer, Baake et al , 1990)

The elements of norm 1 of the usual quaternion ring

$$\mathbb{H} = \mathbb{R} + \mathbb{R}i + \mathbb{R}j + \mathbb{R}k$$

form a group isomorphic to SU_2. Since this group is a 2-fold cover of the orthogonal group SO_3, in particular it contains 2-fold covers of the icosahedral group. One such example is the following list I of 120 vectors:

$$\frac{1}{2}(\pm 1,\, \pm 1,\, \pm 1,\, \pm 1),\ (\pm 1,\, 0,\, 0,\, 0) \text{ and all permutations,}$$

$$(0,\, \pm 1,\, \pm \tau',\, \pm \tau) \text{ and all even permutations,}$$

where $\tau = (1+\sqrt{5})/2$ is the Golden ratio and ' indicates the conjugation map $\sqrt{5} \mapsto -\sqrt{5}$.

The subring $\mathbb{I}$ generated by this group is called the *icosian ring*. The form of the points of I makes it clear that I is a $\mathbb{Z}[\tau]$-module. We let * denote the mapping on I that conjugates each of the coordinates with respect to the unique Galois non-trivial automorphism on $\mathbb{Z}[\tau]$ (defined by sending $\sqrt{5} \mapsto -\sqrt{5}$). Note that $\mathbb{I}^* \neq \mathbb{I}$.

The ring $\mathbb{I}$ is of rank 4 over $\mathbb{Z}[\tau]$ and rank 8 over $\mathbb{Z}$. We make an explicit embedding of $\mathbb{I}$ as a lattice $\tilde{\mathbb{I}}$ in $\mathbb{R}^8$ by the mapping $x \mapsto (x, x^*)$.

This already provides the framework of a cut and project scheme:

$$\begin{array}{ccccc} \mathbb{R}^4 & \xleftarrow{\pi_1} & \mathbb{R}^4 \times \mathbb{R}^4 & \xrightarrow{\pi_2} & \mathbb{R}^4 \\ & & \cup & & \\ & & \tilde{\mathbb{I}} & & \end{array}$$

with the projections being given by the first and second components of (x, x^*).

Remarkably the lattice $\tilde{\mathbb{I}}$ has an entirely natural interpretation as the root lattice of type E_8. Now we wish to show that this cut and project scheme respects the symmetry that is inherent in its construction. Geometrically the points of I form the vertices of a regular polytope P in 4-space and also form the vectors of a root system Δ_4 of type H_4. The Coxeter group H_4 is none other than the group of automorphisms of P (and also of Δ_4), and is in fact very easily described: it is the set of all (14400) maps

$$x \mapsto uxv; \quad x \mapsto u\overline{x}v$$

where $u, v \in I$.

The subgroup of these transformations in which $v = u^{-1}$ is obviously a copy of the icosahedral group and this subgroup stabilizes the 3-dimensional space $\mathbb{R}i+\mathbb{R}j+\mathbb{R}k$ of pure quaternions. These maps provide automorphisms of the rings $\tilde{\mathbb{I}}$ and, via conjugation, on $\tilde{\mathbb{I}}^*$ too, and thus give rise to an action of I as automorphisms on the entire cut and project scheme. If the window W is chosen to be invariant under I then the resulting model set is also I-invariant. Restricting everything to the pure quaternions we get a new cut and project scheme based on the 6-dimensional root lattice D_6 and an icosahedral symmetry.

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