On some generic properties of the finite dimensional nilpotent class 2 Lie algebras

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Denote by $\mathfrak{N}_2$ the set of all finite dimensional nilpotent class 2 Lie algebras over an algebraically closed field K.

Consider a Lie algebra $L \in \mathfrak{N}_2$. Let S be the commutator subalgebra $[L, L]$, $\{z_1, \ldots, z_t\}$ a basis of S, and V a complementary subspace to S of dimension n, i.e. $L = V \oplus S$. Then the product of two elements $x = \bar{x} + \bar{\bar{x}}$ and $y = \bar{y} + \bar{\bar{y}}$ of L with $\bar{x}, \bar{y} \in V$ and $\bar{\bar{x}}, \bar{\bar{y}} \in S$ has the form

$$[x, y] = \varphi_1(\bar{x}, \bar{y}) z_1 + \cdots + \varphi_t(\bar{x}, \bar{y}) z_t \tag{1}$$

for some t-tuple of alternating bilinear forms $\Phi = \Phi(L) = (\varphi_1, \ldots, \varphi_t)$ on V.

On the other hand, given vector spaces S and V over K, a basis $\{z_1, \ldots, z_t\}$ of S, and a t-tuple $\Phi = (\varphi_1, \ldots, \varphi_t)$ of alternating bilinear forms on V, one can define the product of two elements of $L = L(\Phi) = V \oplus S$ by (1). Obviously, $L(\Phi) \in \mathfrak{N}_2$ and S is a central subalgebra of $L(\mathit{\Phi})$.

It is easy to show that the alternating bilinear forms of t-tuple $\mathit{\Phi}(L)$ corresponding to $L \in \mathfrak{N}_2$ are linearly independent. And conversely, if the forms of t-tuple $\mathit{\Phi}(L)$ are linearly independent and $L(\mathit{\Phi}) = V \oplus S$ is a corresponding Lie algebra, then $S = [L(\mathit{\Phi}), L(\mathit{\Phi})]$.

Now denote by B_n^t the set of all t-tuples of alternating bilinear forms on an n-dimensional linear space V over K. Put $M_{n,t} = \{(\varphi_1 : \cdots : \varphi_t) \in \mathbb{P}(B_n^t) \mid \varphi_1, \ldots, \varphi_t \text{ are lineary independent}\}$. $M_{n,t}$ is a Zariski-open subset of the projective space $\mathbb{P}(B_n^t)$.

Definition 1. We say that a generic Lie algebras $L \in \mathfrak{N}_2$ with $\dim L/[L, L] = n$ and $\dim[L, L] = t$ has a property $\mathcal{A}$ if the set of point of $M_{n,t}$ corresponding to the algebras without the property $\mathcal{A}$ contains in some proper Zariski-closed subset.

Since $M_{n,t}$ is irreducible, the dimension of any its closed subset is strictly less than the dimension of $M_{n,t}$.

Theorem 1. *Any Lie algebra $L \in \mathfrak{N}_2$ with $\dim L/[L, L] = n$ and $\dim[L, L] = t > 1$ contains an abelian subalgebra of dimension $s = [\frac{2n+t^2+3t}{t+2}]$. A generic Lie algebra $L \in \mathfrak{N}_2$ with $\dim L/[L, L] = n$ and $\dim[L, L] = t > 1$ doesn't have any abelian subalgebra of dimension $s + 1$.*

Definition 2. Denote by $|\mathcal{A}(n)|$ the number of integers t such that a generic Lie algebra $L \in \mathfrak{N}_2$ with $\dim L/[L,L] = n$ and $\dim[L,L] = t$ has a property $\mathcal{A}$ $\left(1 \leqslant t \leqslant \frac{n(n-1)}{2}\right)$. We say that $\mathcal{A}$ is a generic property of $\mathfrak{N}_2$ if $\frac{|\mathcal{A}(n)|}{\frac{n(n-1)}{2}} \to 1$ as $n \to \infty$.

Theorem 2. *The following properties of $\mathfrak{N}_2$ are generic:*

1) *A generic Lie algebra $L \in \mathfrak{N}_2$ cannot be homomorphically mapped onto any non-abelian Lie algebra of a given dimension N;*

2) *The center of a generic Lie algebra $L \in \mathfrak{N}_2$ coincides with $[L,L]$.*