Предисловие

Представленны слайды 5 лекций по генерическим теориям логики первого порядка и размерным функциям, прочитанных автором на молодежной школе «Алгоритмические вопросы теории групп и смежных областей», Эрлагол 2012. Содержание трех из них основано на статьях автора лекций, написанных совместно с А.Боровиком, А.Мясниковым, Е.Френкель и А.Рыбаловым:

1. A.V. Borovik, A.G. Myasnikov, V.N. Remeslennikov, "Multiplicative measures on free group"// Algebra and Computation, Vol. 13, No. 6, 2003, pp. 705–731;
2. V.N.Remeslennikov, A.N.Rybalov. Dimensional functions over partially ordered sets // in preparation;
3. А. Г. Мясников, В. Н. Ремесленников, Е. В. Френкель, «Свободные произведения групп с объединением: стратификация множеств нормальных форм и оценки», Фундамент. и прикл. матем., 16:8 (2010), 189–221;

и книгах, указанных в списке литературы. Материал по генерическим теориям и (0,1)-закону заимствован из нескольких обзоров и книг на эту тему, указанных в списке литературы. Материал по моделям роста интернета взят из обзора

- А.М. Райгородский. Модели случайных графов и их применения // ТРУДЫ МФТИ. - 2010. - Том 2, №4, С.130–140.

Теория псевдоконечных полей, в виду технических сложностей материала, только обозначена в этих лекциях. Для тех, кто хотел бы познакомиться с этой теорией я рекомендую статью

- J.Ax, The elementary theory of finite fields // Annals of Math., 88 (1968), 239–271.

и статьи Ю.Л.Ершова 60-х годов прошлого столетия.

Generic model theory and Zero-One law for graphs, groups, fields

Vladimir N. Remeslennikov

Motivation

In my series of lectures I define three new notions:

1. *pseudo-theory* of a series of finite algebraic structures;
2. *generic theory* of a series of finite algebraic systems;
3. *dimensional function* over a partially ordered set.

It will be shown that the definitions above are quite useful in mathematics and applications.

Three series of sinite structures and their theories

1. Let FG be the series of finite graphs, T_{fg} the theory of finite graphs in the language $L_g = \{E(x,y)\}$.
2. Let FF be the series of finite fields and T_{ff} the theory of finite fields in the language $L_{ring} = \{+, -, \cdot, 0, 1\}$.
3. Let FGr be the series of finite groups and T_{gr} the theory of finite groups in the language $L_{gr} = \{\cdot, ^{-1}, 1\}$.
4. Concrete series of graph models of Internet growth.

Main aims of my series of lectures

1. We study general properties which have all structures in a given series.
2. We study not arbitrary properties of structures, but only expressible by first-order formulas.
3. We proof that the general properties of a series maybe formalized as first-order theories in corresponding languages.

It will be considered two types of such theories. Any theory of the first type is called *pseudo-finite*. A theory of the second type is called *generic*. The last type o theory depends on the choice of measure over class of structures.

Time for exercises

Famous facts

Graphs:

1. $T_{fg} > T_g$, where T_g is the theory of all graphs.
2. The theories T_{fg} and T_g are algorithmically undecidable (Lavrov, 1965).

Fields:

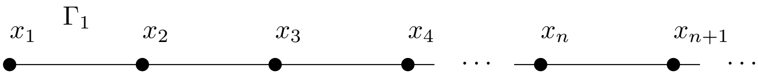
1. $T_{ff} > T_f$, where T_f is the theory of all fields.
2. The theory T_{ff} is decidable (Ax, 1967), but T_f is not.

Groups:

1. $T_{fgr} > T_{gr}$, where T_{gr} is the theory of all groups.
2. Both T_{fgr}, T_{gr} are undecidable.

The proof of strict inclusions of the theories

Graphs: let θ be the sentence: "exactly one vertex has a degree 1, but another ones have degrees 2".

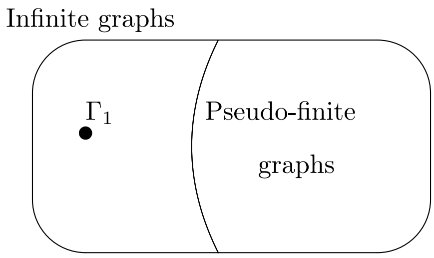


The sentence $\neg\theta$ holds for all finite graphs. **Fields:** let θ be the sentence "for any natural n there exists a unique extension of a degree n". The sentence θ holds for each finite field. **Groups:** A counter-example maybe constructed by any finitely defined but not residually finite group (such groups exist).

Pseudo-finite graphs (fields, groups)

Any infinite model of the theory T_{fg} (resp. T_{ff}, T_{fgr}) is called a *pseudo-finite graph (resp. field, group).*

Theorem. (Follows from Compactness theorem)
The ultraproduct of an infinite number of pairwise non-isomorphic finite graphs (fields, groups) is a pseudo-finite graph (resp. field, group).



Pseudo-finite theories for graphs, fields, groups

For graphs: a small information about it.

For groups: see above.

For fields. The theory PT_{ff} is well-studied. The main results was proved by J. Ax.

1. There exists a useful (recursive) system of axioms of PT_{ff}.
2. PT_{ff} is decidable, it follows the decidability of T_{ff}.
3. There exists a nice classification of complete theories for PT_{ff}.
4. It were obtained famous results in algebraic geometry derived from the facts above.

Extensions toward generic theory

As the extension of $T(\mathcal{K})$ to $PT(\mathcal{K})$ for a series $\mathcal{K}$ is inefficiently in general case, there exists an idea to extend $T(\mathcal{K})$ toward the generic theory $GT(\mathcal{K}, \mu)$ which is defined by a probability measure over $\mathcal{K}$.

Let L be a language, φ a sentence of L, $\mathcal{K}$ a class of L-structures, μ a probability measure defined at measurable subsets from $\mathcal{K}$. Denote $\mathcal{K}(\varphi) = \{A \in \mathcal{K} \,|\, A \models \varphi\}$. Define that φ is *almost surely true with respect to the measure* μ if $\mathcal{K}(\varphi)$ is μ-measurable and $\mu(\mathcal{K}(\varphi)) = 1$.

The set of all almost surely true sentences with respect to the measure μ is called the *μ-generic theory* for a class $\mathcal{K}$ and denoted by $\mathbf{GT}(\mathcal{K}, \mu)$.

Zero-One law for series $\mathcal{K}$

The theory $\mathbf{GT}(\mathcal{K},\mu)$ is always consistent, and its models are called *μ-generic.*

Moreover if theory $\mathbf{GT}(\mathcal{K},\mu)$ is complete (i.e. for any sentence φ either $\mu(\mathcal{K}(\varphi)) = 0$ or $\mu(\mathcal{K}(\varphi)) = 1$), it is said that the class $\mathcal{K}$ satisfies the *Zer-One law* with respect to μ.

Random graphs. Erdõsh-R*é*nyi model

Let us fix a natural n and real $p,\ 0 < p < 1$. Let $X_n = \{1, \dots, n\}$ and G_n be the class of all graphs with the vertex set X_n. Denote by $E(X_n)$ the set of all edges between the vertices from X_n.

Let us define a probability space $G(n,p)$ over the set G_n by the following. Let $p(1_e) = p$ be the probability of the appearance of an edge $e \in E(X_n)$ in the random graph. Hence, $p(0_e) = 1-p = q$ is the probability of the absence of an edge e in the random graph.

Let us fix $Y \subset E(X_n)$. The probability of such type of graph equals

$$p(Y) = p^m q^{C_n^2 - m}.$$

This random model (the binomial distribution with C_n^2 experiments in fact) of graphs over G_n was offered by Erdõsh in 1959.

Zero-One law for Erdõsh-R*é*nyi model

Let Y be a subset of $\mathcal{F}G$ and $Y_n = G_n \cap Y$. If the limit

$$\lim_{n\to\infty} \frac{|Y_n|}{|G_n|}$$

exists let us call it a limit probability of an event Y and denote by $p_\infty(Y)$.

Theorem

For any sentence ϕ in graph theory the limit above exists and

$$p_\infty(\phi) \in \{0,1\}.$$

Moreover,

$$p_\infty(\phi) = 1 \Longleftrightarrow p_\infty(\neg\phi) = 0.$$

Corollary

In the Erdõsh-Rényi model the generic theory of finite graphs $\mathbf{GT}_{fg}$ is complete, and therefore Zero-One law holds.

Questions

Question 1.

The conclusion of the existence of the Zero-One law at the previous slide is not obvious. Indeed, the definition of generic model deals with a probability measure defined on finite graphs, however in the Erdõsh-Rényi model the probability is defined only for graph classes G_n, $n = 1, 2, \dots$

Question 2.

Is it possible define a probability space for the set $\mathcal{F}G$ such that the induced measure over G_n coincides with Erdõsh-R*é*nyi measure?

Answers: both positive

One can define a probability such that $p_\infty = \mu(\mathcal{F}G(\phi))$, where ϕ is a sentence of graph theory, and μ is a probability measure on $\mathcal{F}G$. Let $X_\infty = \{1, 2, \dots, k, \dots\}$ and choose $0 < p < 1$. Further, for any finite disjoint subsets $S, T \subseteq E(X_\infty)$ denote

$$Cone(S,T) = \{Y \subseteq E(X_\infty) | S \subseteq Y, Y \cap T = \emptyset\}.$$

Put $\mu(Cone(S,T)) = p^{|S|}(1-p)^{|T|}$. By the Kolmogorov‘s theorem, μ extended to a probability measure. Moreover, the σ-algebra of μ-measurable sets is generated by all cones конусами $Cone(S,T)$. It is clear that the restriction of μ from X_∞ to $X_n = \{1, \dots, n\}$ induces the Erdõsh-Rényi model on G_n.

Axioms of the theory $\mathbf{GT}_{fg}$ in the Erdõsh-R*é*nyi model

Let $X_m = \{x_1, \dots, x_m\}$ and $Y_n = \{y_1, \dots, y_n\}$ be finite sets of letters and $\psi_{n,m}$ the next sentence of graph theory

$$\forall X_m \forall Y_n \exists z\, (\bigwedge_{i,j}(x_i \neq y_j) \to (\bigwedge_i E(x_i, z) \wedge \bigwedge_j \neg E(y_j, z)))$$

Theorem

In the denotations above we have

1. $p_\infty(\psi_{m,n}) = 1$, therefore the sentences $\psi_{m,n}$ hold in all generic models of the theory $\mathbf{GT}_{fg}$;
2. $\{\psi_{n,m} | (n,m) \in \mathbb{N}^2\}$ is the system of axioms for $\mathbf{GT}_{fg}$.

Properties of $\mathbf{GT}_{fg}$ in the Erdõsh-Rényi model

Theorem
For the theory $\mathbf{GT}_{fg}$ in the Erdõsh-Rényi model the following holds:

- $\mathbf{GT}_{fg}$ is countably categorical;
- $\mathbf{GT}_{fg}$ is decidable.

As the theory $\mathbf{GT}_{fg}$ is countably categorical, there exists a unique (up to ismorphism) generic graph in the Erdõsh-Rényi model. It is Rado graph.

The construction of Rado graph
Let $\Gamma_0 = \Gamma$be an arbitrary graph (not necessary finite). $\mathbf{V}(\Gamma) = X_0 \neq \emptyset$ and $\mathcal{F}(X_0) = \{s | s \text{is a finite subset in } X_0\}$, $Z = \{z_s| \text{ for } s \in \mathcal{F}(X_0)\}$. Define the new graph Γ_1 by

$$X_1 = \mathbf{V}(\Gamma_1) = X_0 \cup Z.$$

$$E(\Gamma_1) = E(\Gamma_0) \bigcup_{s \in \mathcal{F}(X_0)} E(z_s, s),$$

where the set $E(z_s, s)$ consists of edges which joint the vertex z_s with vertices from the set s.
Thus, $\Gamma_0 < \Gamma_1$.
Let us iterate this operation and obtain the limit graph

$$\Gamma_\infty = \bigcup_{i=0}^{\infty} \Gamma_i.$$

Proposition
The graph Γ_∞ is a model of the theory $\mathbf{GT}_{fg}$.

Properties of Rado graph

1. ultra-transitivity, i.e. it is k-transitive for any natural k;
2. let H be an arbitrary finite graph, then Rado graph contains infinitely many copies of H;
3. the diameter of Rado graph equals 2.

Generic model theory and Zero-One law for graphs, groups, fields

Vladimir N. Remeslennikov

Generic theory for partial orders
Fix a language $L = \{\leq\}$. Write axioms:

1. $\forall x \; (x \leq x)$;
2. $\forall x, y, z \; (x \leq y \wedge y \leq z \to x \leq z)$;
3. $\forall x, y \; (x \leq y \wedge y \leq x \to x = y)$.

Any structure satisfying the axioms above is called a *partial order*.

Surprise!

Really,
the formula "there no chains of length greater than three"

$$\forall x_0, x_1, x_2, x_3 \; (\bigwedge_{0 \leq i \leq 2} x_i \leq x_{i+1} \to \bigvee_{0 \leq i \leq 2} x_i = x_{i+1})$$

belongs to the generic theory!!! See Kleitman D.J., Rothschild B.L. "Asymptotic enumeration of partial orders on a finite set" for the proof.

It follows that every model (partial order) of the generic theory has at most three levels: L_0, L_1, L_2.

Zero-One Law for graphs and partial orders has the similar proofs

Plan of the proof

1. write extension axioms;
2. prove that any axiom has the measure 1;
3. prove that the generic theory is countable categorical;
4. from the facts of model theory it follows that the generic theory is complete, and Zero-One law holds.

Extension axioms

Note that the relation $x \in L_i$ is expressible by first order formula.

1. For any $j, k, l \geq 0$ there is an axiom saying that for all distinct $x_0, \ldots, x_{j-1}, y_0, \ldots, y_{k-1}$ from L_1 and all distinct $z_0, \ldots, z_{l-1} \in L_0$ there exists $z \in L_0$ not equal to $z_0, \ldots, z_{l-1}$ such that
$$\bigwedge_{i<j} z \leq x_i \wedge \bigwedge_{i<k} z \not\leq y_i.$$
2. For any $j, k, l \geq 0$ there is an axiom saying that for all distinct $x_0, \ldots, x_{j-1}, y_0, \ldots, y_{k-1}$ from L_1 and all distinct $z_0, \ldots, z_{l-1} \in L_2$ there exists $z \in L_2$ not equal to $z_0, \ldots, z_{l-1}$ such that
$$\bigwedge_{i<j} z \geq x_i \wedge \bigwedge_{i<k} z \not\geq y_i.$$

The last extension axiom

3 For any $j, j', k, k', l \geq 0$ there is an axiom saying that for all distinct $x_0, \ldots, x_{j-1}$ and $y_0, \ldots, y_{k-1}$ from L_0, all distinct $x'_0, \ldots, x'_{j-1}$ and $y'_0, \ldots, y'_{k-1}$ from L_2, and all distinct $z_0, \ldots, z_{l-1} \in L_1$ there exists $z \in L_1$ not equal to $z_0, \ldots, z_{l-1}$ such that
$$\bigwedge_{i<j} x_i \leq z \wedge \bigwedge_{i<k} y_i \not\leq z \wedge \bigwedge_{i<j'} z \leq x'_i \wedge \bigwedge_{i<k'} z \not\leq y'_i.$$

Time for thinking

II. Generic models in the class of sparce graphs

Let $\lambda > 0$ be a real number. Denote $\mathcal{K}_\lambda = \{\Gamma \in \mathcal{F}G \, || V(\Gamma)| \geq \lambda |E(\Gamma)|\}$. $\mathcal{K} \subset \mathcal{F}G$ is called the set of sparse graphs if there exists a real $\lambda > 0$ such that $\mathcal{K} \subseteq \mathcal{K}_\lambda$.

In the classes of sparse graphs the Erdõsh-Rényi model is not useful. For this reason it was develop another probability models for the class $\mathcal{K}_\lambda(n),\ n = 0, 1, 2, \ldots$

In these models the probability does not depend on n and satisfies the power-law:
$$p_n = \frac{C}{n^\lambda},$$
where the constant C gives that the sum of probabilities of all graphs from $\mathcal{K}_\lambda(n)$ equals 1.

The Shelah-Spencer model (1988)

Fix an irrational number $0 < \lambda < 1$. Let $\mathcal{K}_\lambda(n)$ be the set of all λ-sparse graphs with n vertices and $p_n = \frac{C}{p^\lambda}$. Denote by T^λ the generic theory for $\mathcal{K}_\lambda$.

Theorem (Shelah, Spencer, 1988)

For an irrational number $0 < \lambda < 1$ the generic theory T^λ is complete and, therefore, satisfies the Zero-One law.

Properties of Shelah-Spencer model

1. The proof of is based on the next classical result in number theory: the set $M(\lambda) = \{a - \lambda b | a, b \in \mathbb{N}^+\}$ is dense in reals;
2. there exists a useful $\forall\exists\forall$ - axiomatization of the theory T^λ;
3. T^λ has the dimensional order property (DOP).

Question

Is the generic theory $GTh(\mathcal{K})$ simple (e.g. stable) for any class $\mathcal{K}$ from $\mathcal{K}_\lambda$, $0 < \lambda < 1$.

Answer

No. One can construct a hard (unstable) generic model by Hrushovski's constructions. For more details see the prepint Justin Brody "On generic graphs with intrinsic transcendentals".

Observation (due to Barabasi-Albert)

One can consider the Internet as a graph, where the vertices are cites and directed edges are links. Obviously, this graph (so-called a *web-graph*) is directed, has multiply edges and loops. One can observe that such graph has the next properties:

1. the web-graph is strongly sparse, i.e. it has only kt edges for t vertices, where $k \geq 1$ is a constant;
2. the diameter of the web-graph is only 5–7;
3. the empiric probability that a vertex has a degree d equals c/d^{λ}, where $\lambda \approx 2.1$, c is the normalizing coefficient;
4. if there appears a new cite, it tries to get a link to a popular ones.

Thus, the Erdosh–Renyi model is useless for such graph.

The Bollobas-Riordan model

Let us construct the web-graph in the Bollobas-Riordan model.

Firstly, we define a sequence of random graphs $\{G_1^n\}$, where the n-th graph has exactly n edges and n vertices. Further, we obtain the sequence $\{G_k^n\}$, where the graph G_k^n consists of n vertices and kn edges.

Put that G_1^1 has a vertex 1 and a loop $(1,1)$. Assume the graph G_1^{n-1} is constructed, it has the vertices $\{1, 2, \ldots, n-1\}$ and $n-1$ edges. Add a vertex n and an edge (n, i) for some vertex i. A loop (n, n) appears with the probability $\frac{1}{2n-1}$, an edge (n, i) appears with the probability $\frac{deg\, i}{2n-1}$, where $deg\, i$ is the degree of the vertex i. Such random process is well-defined, as

$$\sum_{i=1}^{n-1} \frac{deg\ i}{2n-1} + \frac{1}{2n-1} = \frac{2n-2}{2n-1} + \frac{1}{2n-1} = 1.$$

Now we define the graph G_k^n

Take G_1^{kn}. It is a graph with kn vertices and kn edges. Divide the set of vertices into the parts of a size k:

$$\{1, \ldots, k\},\ \{k+1, \ldots, 2k\},\ \ldots,\ \{k(n-1)+1, \ldots, kn\}.$$

Each piece collapses into a single vertex preserving the edges, i.e. if there are edges between vertices in one part, we obtain multiply loops. The edges connecting different parts become multiply edges in the new graph.

Iterate this procedure, we obtain the web-graph.

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Books:

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- any book devoted to finite model theory (e.g. see L.Libkin "Elements of finite model theory", Springer, 326p, 2012);
- R. Diestel "Graph theory", Springer, 451p, 2012 (zero-one law for graphs).

Dimensional functions over partially ordered sets

Vladimir N. Remeslennikov

Motivation: Classical origins

Let F be a field. $IrAlS = \{Y \subset F^n,\ Y$ is algebraic set$\}$.

Definition

$$dim(Y) = \max\{l \in \mathbb{N} : Y_l \subset Y_{l-1} \subset \ldots \subset Y_1 \subset Y_0 = Y,$$
$$\text{where } Y_i \text{ irreducible algebraic set}\}.$$

Question

Is there a good definition of dimension for arbitrary algebraic structure?

Answer

Yes. There exist good definitions if algebraic structure is equationally noetherian.

Motivation: Origins from combinatorial algebra

Let A be an alphabet, A^* be the free monoid and $F(A)$ be the free group. Most of problems from discrete optimization and algorithmical group theory can be formulated in terms of A^* and $F(A)$.

Question
Let $R \subset A^*$ or $R \subset F(A)$. Is there a good definition of dimension for such R?

Answer
Yes. If R is regular set.

Motivation: Origins from model theory

Question
How to define a measure for series $\mathcal{K}$ of finite algebraic structures such that $GTh(\mathcal{K})$ has a good description?

Answer
Sometimes it is possible. And more the better such examples!

Partially ordered sets

Definition
A **partial order** is a binary relation $\leq$ over a set M such that

- $\forall a \in M \; a \leq a$ (reflexivity);
- $\forall a, b \in M \; a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry);
- $\forall a, b, c \in M \; a \leq b$ and $b \leq c$ implies $a \leq c$ (transitivity).

Definition
Set M with a partial order is called a **partially ordered set (poset)**.

Linearly ordered abelian groups

Definition
A set A equipped with addition $+$ and a linear order $\leq$ is called **linearly ordered abelian group** if

1. $\langle A, + \rangle$ is an abelian group;
2. $\langle A, \leq \rangle$ is a linearly ordered set;
3. $\forall a, b, c \in A \; a \leq b$ implies $a + c \leq b + c$.

Definition
The semigroup A^+ of all nonnegative elements of A is

$$A^+ = \{a \in A \mid 0 \leq a\}.$$

Dimensional functions

Let M be a poset and A be a linearly oredered abelian group.

Definition
Function $d : M \to A^+$ is called A**-dimensional function** over M if

1. $\forall x, y \in M$ if $x < y$ in M, then $d(x) < d(y)$ in A.
2. $\forall x, y \in M \; \exists x', y'$ such that if $d(x) < d(y)$, then $d(x) \leq d(x')$, $d(y') \leq d(y)$ and $x' < y'$.

Definition
A-dimensional function over M is called **strongly dimensional function** if

1. $\forall x, y \in M \; \exists x', y'$ such that if $d(x) < d(y)$, then $d(x) = d(x')$, $d(y) = d(y')$ and $x' < y'$.

Dimensional functions: axioms

Lemma
Let a function $d : M \to A^+$ satisfies only the first axiom of dimensional function:

1. $\forall x, y \in M$ if $x < y$ in M, then $d(x) < d(y)$ in A.

Then there exists a function $d' : M \to A^+$, satisfying both axioms, that is dimensional function.

Flows

Dimensional function $d : M \to A^+$ defines equivalence $\tilde{d}$ over M in the following way

$$m_1 \sim_d m_2 \leftrightarrow d(m_1) = d(m_2)$$

and is a homomorphism in the category of posets. Let us $[m_1] \leq_d [m_2] \leftrightarrow d(m_1) \leq d(m_2)$.

Fact
$M/\sim_d$ is linearly ordered set.

Definition
$M/\sim_d$ is called d-**flow**. Order type of d-flow is denoted by $\pi_d(M)$.

Definition
Element $d(m)$ from A is called A-**dimension** of m.

Equivalence of dimensional functions

Let M be an poset and d_1, d_2 are dimensional functions over M with values in some linearly ordered abelian groups.

Definition
Dimensional functions d_1, d_2 are **equivalent** ($d_1 \sim d_2$) if order types $\pi_{d_1}(M)$ and $\pi_{d_2}(M)$ are isomorphic.

Ordinal dimensional functions

Definition
Dimensional function $d : M \to A^+$ is called **ordinal** if $\pi_d(M)$ is a well ordered set.

Definition
Poset M is called a **set of ordinal type** if there exist a strongly dimensional function for it.

Dimensional functions over Artin sets

Definition
Poset M is called **Artin set** if any chain $a_1 > a_2 > \dots$ in M is finite.

Theorem

1. Every Artin set is a poset of ordinal type.
2. For every Artin set there exists unique up to equivalence strongly dimensional function.

Corollary
Let M be a finite poset. There exists a dimensional function over M and every two such functions are equivalent.

Non-equivalent dimensional functions

$L_1 = \{[0,1], 2, 3, [4,5]\}$, $L_2 = \{[6,7], 8, [9,10]\}$ with natural order. $M = L_1 \cup L_2$. Any $a \in L_1$ and $b \in L_2$ are not comparable.

Proposition
There are two non-equivalent dimensional functions over M.

$$d_1(x) = \begin{cases} x, & x \in L_1, \\ x - 1, & x \in L_2. \end{cases}$$
$$d_2(x) = \begin{cases} x, & x \in L_1, \\ x - 10, & x \in L_2. \end{cases}$$
$\pi_{d_1}(M) = \{[0,1], 2, 3, [4,6], 7, [8,9]\}$. $\pi_{d_2}(M) = \{[-4,-3], -2, [-1,1], 2, 3, [4,5]\}$.

Existence of dimensional functions

Theorem
For every poset M there exists a linearly ordered abelian group A and dimensional function $d : M \to A^+$.

Corollary
For every poset M there exists a linearly ordered field F and dimensional function $d : M \to F^+$.

Question
Is it true that for every poset there exists *strongly* dimensional function?

Geometric dimensional functions

Theorem
Let $d_1 : M_1 \to A$ be a dimensional function over poset M_1 and $d_2 : M_2 \to A$ be a dimensional function over poset M_2. Then the function $d : M_1 \times M_2 \to A$ defined as

$$\forall m_1 \in M_1 \ \forall m_2 \in M_2 \ d((m_1, m_2)) = d_1(m_1) + d_2(m_2)$$

is dimensional function over $M_1 \times M_2$.

Lattice dimensional functions

Let M be a lattice and A be a linearly ordered abelian group.

Definition
Function $d : M \to A^+$ is called lattice A-dimensional function if

1. d is A-dimensional function,
2. $\forall x, y \in M \ d(x \vee y) + d(x \wedge y) = d(x) + d(y)$.

Theorem
Let M be locally finite semimodular lattice. Then there exists a linearly ordered abelian group A and lattice A-dimensional function over M.

Corollary
Let M be distributive (Boolean) lattice. Then there exists a linearly ordered abelian group A and lattice A-dimensional function over M.

Lattice dimensional functions over interval Boolean lattice

Question
Let S be countable set and $P(S)$ be the Boolean lattice of subsets of S. When every two dimensional functions over $P(S)$ are equivalent?

Application: Universal algebraic geometry

Let $\mathcal{B}$ be an algebraic structure in a functional language and $Y \subseteq B^n$ is a non-empty subset. Denote by $\mathbb{I}rr(Y)$ the family of all irreducible algebraic over $\mathcal{B}$ subsets of Y.

If $\mathcal{B}$ satisfies the *Unification theorems* (E.Daniyarova, A.Miasnikov, V.Remeslennikov), then $\mathbb{I}rr(Y)$ is poset and we can define an ordinal dimensional function *dim* over $\mathbb{I}rr(Y)$.

Ordinal dimensional function *dim* let us to use induction to build universal algebraic geometry over $\mathcal{B}$.

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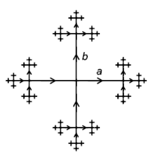
Asymptotic classification of subsets of $F(X)$

Vladimir N. Remeslennikov

Graphs

Consider a Cayley graph $C(F, X)$ for a free group F generated by finite set X, let $|X| = m$.



Pic. 1. Cayley graph for $F = F(a, b)$.

Generation of random words in $F(X)$

Let $F = F(X)$ be a free group with a basis $X = \{x_1, \ldots, x_m\}$. We use the following no-return random walk on the Cayley graph of F with respect to the generating set X. We start at the identity element 1 and either do nothing with probability $s \in (0, 1]$, or move to one of $2m$ adjacent vertices with equal probabilities $(1 - s)/2m$. If we are in a vertex $v \neq 1$, we either stop at v with probability s (and return the value v as the output), or move with the probability $\frac{1-s}{2m-1}$ to one of the $2m - 1$ adjacent vertices lying away from 1, thus producing a new freely reduced word $vx_i^{\pm 1}$.

It is easy to see that the probability $\mu_s(w)$ for our process to terminate at a word w is given by the formula

$$\mu_s(w) = \frac{s(1-s)^{|w|}}{2m(2m-1)^{|w|-1}},$$

and

$$\mu_s(1) = s.$$

Hence the length of words produced by our process are distributed according a geometric law.

Probability measures of $R \subseteq F(X)$

Let $R \subseteq F(X)$. Denote by $n_k(R) = |R \cap S_k|$ and by $f_k(R)$ the relative frequencies

$$f_k = \frac{n_k}{|S_k|},$$

where $S_k = \{w \| w| = k\}$ is a sphere of the radius k. Put

$$\mu_s(R) = \sum_{w \in R} \mu_s(w)$$

or equivalently

$$\mu_s(R) = s \sum_{k=0}^{\infty} f_k (1-s)^k$$

μ_s is an atomic probabilistic measure. Hence, there exists a family of measures $\{\mu_s | s \in (0,1]\}$ defined over the subsets of $F(X)$.

λ-measure for R

Put $s = 0$ and obtain a non-stopping random walk on the Cayley graph $C(F, X)$. In this case the probability $\lambda(w)$ that the walker reaches an element $w \in F$ in $|w|$ steps is equal to

$$\lambda(w) = \frac{1}{2m(2m-1)^{|w|-1}}, \text{ if } w \neq 1, \text{ and } \lambda(1) = 1.$$

This gives rise to an atomic measure

$$\lambda(R) = \sum_{w \in R} \lambda(w) = \sum_{k=0}^{\infty} f_k(R).$$

This measure is not probabilistic (as $\lambda(F) = \infty$). We shall call λ the *frequency measure* on F. If R is λ-measurable (i.e. $\lambda(R) < \infty$) then $f_k(R) \to 0$ when $h \to \infty$, so intuitively, the set R is "small" in F.

Proposition

For any λ-measurable subset $R \subseteq F$ there exists a finite subset $K \subseteq F$ with $\lambda(R) = \lambda(K)$.

The λ-measure is multiplicative:

$$\lambda(u \circ v) = \lambda(u)\lambda(v) \; \forall u, v \in F(X).$$

Generating function of a subset $R \subset F(X)$

Let R be a subset of $F = F(X)$ and S_k is a sphere of radius k in the Cayley graph of $F(X)$.

Denote by $f_k = \dfrac{|R \cap S_k|}{|S_k|}$ so called relative frequencies of R in F.

The generating function for R is a formal series in $R[[t]]$:

$$g_R(t) = \sum f_k t^k.$$

Examples

$g_F(t) = \frac{1}{1-t}$, $g_{wF}(t) = \frac{1}{2m(2m-1)^{|w|-1}} \frac{1}{1-t}$, where $wF = \{w \circ v | v \in F\}$ is a cone.

Cesaro density

For a subset R of F we define the limit measure $\mu_0(R)$ (the Cesaro density):

$$\mu_0(R) = \lim_{s\to 0^+} \mu_s(R) = \lim_{s\to 0^+} s \sum_{k=0}^{\infty} f_k(1-s)^k.$$

The function μ_0 is additive. It is easy to construct a set R such that $\mu_0(R)$ does not exist.

From the theorem by Hardy and Littlewood it follows that μ_0 can be computed as the Cesaro limit

$$\mu_0(R) = \lim_{n\to\infty} \frac{1}{n}(f_1 + \ldots + f_n).$$

Examples

- $F = F_1 = \mathbb{Z}$, $R = 2\mathbb{Z}$;
- H is a subgroup of finite index in $F(X)$, $\mu_0(H) = \frac{1}{|F:H|}$.

Asymptotic classification of subsets

Now we introduce a classification of subsets in F:

- *thick subsets*: $\mu_0(R)$ exists and $\mu_0(R) > 0$;
- *sparse subsets*: $\mu_0(R) = 0$ and $\lambda(R)$ exists;
- *subsets of intermediate density*: $\mu_0(R) = 0$ but $\lambda(R)$ does not exist;
- *singular subsets*: $\mu_0(R)$ does not exist.

Rational and algebraic subsets

Let R be a subset of F. By the definition, its generating function $g_R(t)$ is analytic on $(0,1)$. The subset R is called *rational (algebraic)* if so is $g_R(t)$. We say R is *smooth* if $g_R(t)$ can be analytically extended to a neighbourhood of 1.

Poles

It is well-known fact that singular points of an algebraic function are either poles or branching points. A function $g_R(t)$ has no any singularity at 1 or it has a pole if $g_R(t)$ is rational.

Again, by the Hardy–Littlewood theorem $\mu_0(R) = res_1\, g_R(t)$. We have the formulas:

$$g_F(t) = \frac{1}{1-t},\ \mu_0(F) = 1,$$

$$g_{wF}(t) = \frac{1}{2m(2m-1)^{|w|-1}} \frac{1}{1-t},$$

$$\mu_0(wF) = \frac{1}{2m(2m-1)^{|w|-1}},$$

where $wF = \{w \circ v | v \in F\}$ is a cone.

Why?

Why the function $\frac{1}{(1-z)^2}$ is not generating for some regular subset $R \subseteq F(X)$?

A generation function for regular set has

- either no singularity at 1 (sparse set),
- or a pole of order 1 (thick set).

But $\frac{1}{(1-z)^2}$ has no.

Dimensional functions for regular subsets

Let R be a regular subset from $F(X)$ and $A = \overleftarrow{\mathbb{Q} \times \mathbb{Q}}$ the abelian group with left lexicographical order. For any element of the class $S_{reg} = \{R | R \text{ is regular}\}$ we define a map

$$\dim\colon S_{reg} \to A^+$$

by

$$\dim R = (\mu_0(R), \lambda(R')),$$

where the sparse set R' is effectively constructed from R.

Result

Theorem
The map dim: $S_{reg} \to A^+$ is a dimensional function.

Algorithm for computing of $dim(R)$

1. Compute $g_R(t) = \frac{h(t)}{f(t)}$.
2. If $f(1) \neq 0$, then $\mu_0(R) = 0$ and $\lambda(R) = g_R(1) \Rightarrow dim(R) = (0, \lambda(R))$.
3. If $f(1) = 0$, then $f(t) = (1-t)f_0(t)$ and $f_0(1) \neq 0$.
4. Compute $res_1(g_R(t)) = c$, $0 < c \leq 1$.
5. Find $r(t) = g_R(t) - \frac{c}{1-t}$. Function $\pm r(t)$ is generating function for regular set R', which are from decomposition of automaton $\mathcal{A}(R)$. And the set R' is sparse. Compute $\lambda(R') < \infty$.
6. $dim(R) = (c, \pm\varepsilon\lambda(R'))$.

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Asymptotic classification of regular sets

Vladimir N. Remeslennikov

Asymptotic classification of regular sets

Part I. Regular sets

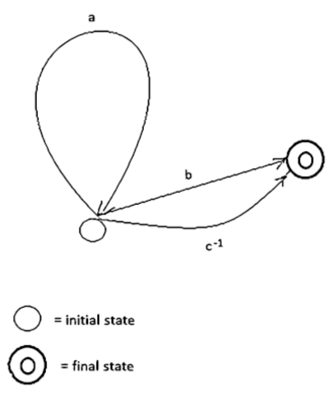
1 Regular sets

Finite automata
Let $X = \{x_1, \ldots, x_n\}$ be a finite alphabet, X^* be the free monoid generated by X and $F(X)$ be the free group generated by X.

A **finite automaton** $\mathcal{A}$ is a finite labelled oriented graph (possibly with multiple edges and loops). We refer to its vertices as states; some states called **initial**, some states called **final**.

Assume that every edge of the graph is labelled by one of symbols $x^{\pm 1}, x \in X$. A **path** in $\mathcal{A}$ is a set of edges $e_0, \ldots, e_n$, such that for each $i = 1, \ldots, l$ the end point e_{i-1} is equal to the starting point of e_i. Reading labels on edges along the path consequently we get the label of the path.

Finite automata



Pic. 1. Finite automaton $\mathcal{A}$.

Facts about regular sets

The **language accepted by $\mathcal{A}$** is a set $L = L(\mathcal{A})$ of labels of paths from an initial state to an final state.

An automaton is said to be **deterministic** if for any state there is at most one arrow with the given label exiting from the state.

A **regular** set over X^* is a language accepted by a finite deterministic automaton.

Finite automata

Example 1. The automaton $\mathcal{A}$ accept the language $L_1 = L_1(\mathcal{A}) = \{a^*b\} \bigcup \{a^*c^{-1}\}$. Therefore, the language L_1 over the alphabet (a, b) is regular.

Example 2. The language $L_2 = \{a^n b^n | n \text{ is a positive integer}\}$ over the alphabet (a, b) is nonregular.

Facts about regular sets

If A and B are regular subsets of F, then

- 1) sets $A \cap B$, $A \cup B$, $A \setminus B$ are regular,
- 2) A^* is regular;
- 3) $\overline{A}$ is regular, where $\overline{A}$ is the **prefix closure** of A, i.e. the set of all initial segments of all words in A;
- 4) $A \circ B$ is regular, where $a \circ b$ means that there is no cancellation between $a \in A$ and $b \in B$.

Adjacency matrix

One can associate with the automaton $\mathcal{A}$ it's adjacency matrix A by taking $n \times n$ matrix and writing the number of arrows from state i to j in the position (i, j), where n is the number of states of $\mathcal{A}$.

Fact 1. The number of different paths of length k from i to j is equal to $(A^k)_{i,j}$.

Fundamental matrix

Set $T = tA$ and let $B = T + T^2 + \ldots$ be the matrix with entries b_{ij} from the ring of formal power series $R[[t]]$.

Then $B(E - T) = T$ and $B = T(E - T)^{-1}$.

The matrix B is called fundamental matrix of $\mathcal{A}$.

Fact 2. Entries b_{ij} of B are rational functions of t.

Generating function of a regular set

Let R be a subset of $F = F(X)$ and S_k is a sphere of radius k in the Cayley graph of $F(X)$.

Denote by $f_k = \dfrac{|R \cap S_k|}{|S_k|}$ so called relative frequencies of R in F.

The generating function for R is a formal series in $R[[t]]$:

$$g_R(t) = \sum f_k t^k.$$

Generating function of a regular set

Theorem 1. *The generating function of a regular set is a rational function of t.*

Proof. Indeed,

$$g_R(t) = \sum_{i \in I, j \in J} b_{ij},$$

where I is the set of all initial states, and J is the set of all final states of $\mathcal{A}$.
Then the result follows from Fact 2. •

$\lambda-$measure

We introduce atomic measure on subsets R of $F = F(X)$, $rank(F) = m < \infty$:

$$\lambda(R) = \sum_{k=0}^{\infty} f_k(R).$$

In particular, if $R = \{w\}$ and $|w| = k$, $k \neq 0$, then $\lambda(R) = \dfrac{1}{2m(2m-1)^{k-1}}$,

It is easy to check that $\lambda(R) = \sum\limits_{w \in R} \lambda(w) = \sum\limits_{k=0}^{\infty} f_k(R)$, i.e. $\lambda(R)$ is an atomic measure indeed. We will call it frequency measure on F.

$\lambda-$measurable sets

We say R is $\lambda-$measurable, if $\lambda(R)$ is finite.

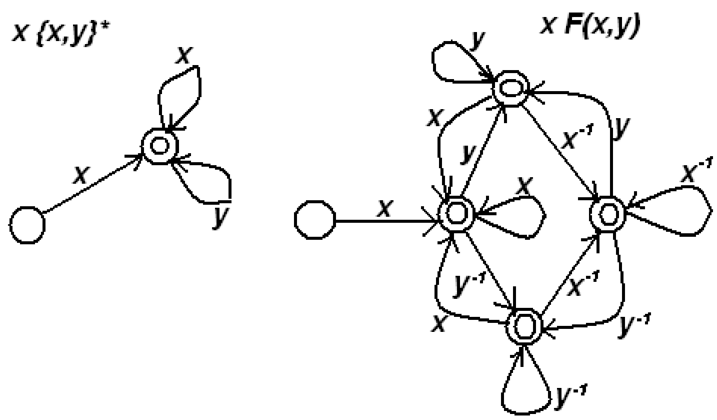
A set R is termed *exponentially $\lambda-$measurable*, if $f_k(R) \leq q^k$ for all sufficiently large k and positive constant $q < 1$.

Fact 3.

- (i) If $R \subseteq F(X)$ is $\lambda-$measurable, then R is negligible. The reverse is not always true.
- (ii) The measure λ is additive, but not σ-additive.

Cones

A cone $C(w)$ with the handle w is a set of all elements in F containing the given word w as initial segment. Obviously, cones are regular sets.



Pic. 2. Finite automaton for cones.

Asymptotic classification of regular sets

Theorem 2. [Borovik, Miasnikov, Remeslennikov, 2003] Let R be a regular subset of $F = F(X)$. Then there are only two possibilities:

i) The prefix closure $\overline{R}$ of R contains a cone, or

ii) R is $\lambda-$measurable.

Asymptotic classification of regular sets

Theorem 2'. [Borovik, Frenkel, Miasnikov, Remeslennikov, 2003, 2005] Let R be a regular subset of $F = F(X)$. Then there are only two possibilities:

i) The prefix closure $\overline{R}$ of R contains a cone, or

ii) R is exponentially $\lambda-$measurable.

Asymptotic classification of regular sets

Part II. Calculating of $\lambda-$measure

2 Calculating of $\lambda-$measure

Discrete-time Finite Markov chains

Let

1) $S = (s_0, s_1, \ldots, s_n)$ be a finite number of states;

2) P be a stochastic transition matrix;

3) $\pi = (q_0, q_1 \ldots, q_n)$ be a stochastic initial vector.

Example of a Markov process

Example 3. Let $S = (s_0, \ldots, s_4)$.

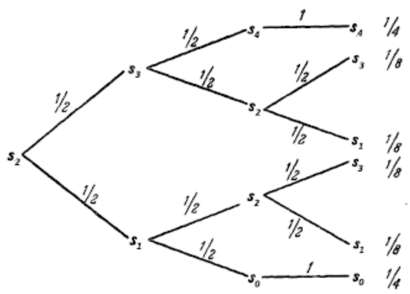
Assume that if the process reaches state s_0 or s_4 it remains there from that time on and let the transition matrix is given by

$$P = \begin{array}{c} \\ s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{ccccc} s_0 & s_1 & s_2 & s_3 & s_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}$$

Track of the process

Let us assume that the process starts in state s_2.

Then $\pi = (0, 0, 1, 0, 0)$ and



Pic.2. Track of the process.

Interpretation of P^k

Fact 4.

i) Powers P^k, $k = 1, 2, \ldots$ of a transition matrix P are stochastic matrices;

ii) entries $p^k_{i,j}$ of P^k represent a probability of a transition from s_i to s_j in k steps.

Binary relation on states

Let $\preceq$ be a binary relation on S. Namely, set $s_i \preceq s_j$ iff the process can go from s_i to s_j (not necessarily in one step).

The states are divided into equivalence classes and let $\sim$ be the corresponding equivalence relation on S, defined by $\preceq$.

Let $[s]$ be a equivalence class with a representative s;

$\overline{S} = S/\!\sim$ be a quotient set of $\sim$ and

"$\leq$" be a partial ordering on $\overline{S}$, induced by $\preceq$.

Classification of states

States from the minimal equivalence classes for the system $<\overline{S}, \leq>$ are called ergodic states.

The remaining states called transient.

It is clear that if the process enters the equivalence class of some ergodic state, it can never leave this class, while if it leave a transient state, it will never return to its equivalence class again.

If the equivalence class of some ergodic state contains only one element, this state is called absorbing.

A canonical form for the transition matrix

Let us renumber states of a Markov chain such that ergodic states ($T-$states) come first, and then come transient states.

In this case P looks like

$$P = \begin{pmatrix} T & 0 \\ R & Q \end{pmatrix}$$

and

$$P^k = \begin{pmatrix} T^k & 0 \\ R' & Q^k \end{pmatrix}$$

Example

For instance, for Example 3 we obtain

$$P= \begin{array}{c} \\ s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \end{array} \begin{array}{c} \begin{array}{ccccc} s_0 & s_1 & s_2 & s_3 & s_4 \end{array} \\ \left(\begin{array}{cc|ccc} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \hline q & 0 & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & p & 0 & q & 0 \end{array}\right) \end{array}$$

Probability theorem

Theorem 3. In any finite Markov chain, no matter where the process starts, the probability after k steps that the process is in an ergodic state tends to 1 as k tends to infinity.

Corollary 1. There are numbers $b > 0, 0 < c < 1$ such that

$$p_{ij}^k \leq bc^k$$

for any transient states s_i and s_j.

Fundamental matrix

A chain, all of whose ergodic states are absorbing, is called an absorbing chain.

Fact 5. For any absorbing Markov chain $\lim_{k\to\infty} Q^k = 0$ and hence $E-Q$ has an inverse and $(E-Q)^{-1} = E+Q+Q^2+\ldots$.

For an absorbing Markov chain we define the fundamental matrix to be

$$N = (E-Q)^{-1} = E + Q + Q^2 + \ldots.$$

Applications of fundamental matrix

Theorem 4. If b_{ij} is the probability of the process to leave a transient state s_i and stop at an absorbing state s_j, then

$$B = \{b_{ij}\} = NR,$$

where R is the block in P and N is the fundamental matrix of a process.

Special automata

A deterministic automaton $\mathcal{A}$ we will call special, if

1.) it has only one initial state v_0 and one final state v_f, $v_0 \neq v_f$;

2.) there are no arrows entering the initial state v_0 and there are no arrows exiting from the final state v_f;

3.) for any state v of $\mathcal{A}$, all arrows which enter v have the same label $x \in X \cup X^{-1}$ and arrows exiting from v cannot have label x^{-1}.

4.) there is at least one state $v \neq v_f, v_0$ such that there is an edge labelled a entering this state but some label from $X \cup X^{-1} \setminus \{a\}$ is not present on arrow exiting from v.

Calculation of $\lambda-$measure for special automata

Theorem 5. Let $\mathcal{A}$ be a special automaton and $L = L(\mathcal{A})$. Then

$$\lambda(L(\mathcal{A})) < 1.$$

Sketch of the proof of Theorem 5.

We form an absorbing Markov chain for $\mathcal{A}$:

S contains all states of $\mathcal{A}$ together with additional state D;

initial distribution

$$\pi = (0 \quad \ldots \quad \underset{v_0}{1} \quad \ldots \quad 0)$$

transition matrix entries

$$p_{ij} = \frac{\text{number of arrows from } s_i \text{ to } s_j}{2|X| - 1}$$

and $p_{DD} = 1$.

Lemma

Lemma 1. Let $\mathcal{A}$ be a special automaton and $L = L(\mathcal{A})$. Then $\lambda(L(\mathcal{A}))$ is a probability of the process to stop in v_f.

Since the probability to stop at D is equal to zero, then $\lambda(L(\mathcal{A})) < 1$.

The End. Thank you!