HOW TO USE THE NORMAL STRUCTURE, WHEN IT IS IMPOSSIBLE TO USE THE NORMAL STRUCTURE

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Lecture 1. Groups of induced automorphisms

$A \leqslant G$ and $A \trianglelefteq G$ mean that A is a subgroup and is a normal subgroup of G respectively.

Assume that G acts on a set X, i.e. a homomorphism $G \to \mathrm{Sym}(X)$ is defined. Elements of X are called points. The image of $x \in X$ under $g \in G$ is denoted by $(x)g$ or x^g.

Symbol G_x denotes the stabilizer of a point x (i.e. $G_x = \{g \in G \mid (x)g = x\}$), while x^G denotes the orbit of x (i.e. $x^G = \{(x)g \mid g \in G\}$).

If $H \leqslant G$, then we are always assuming that G acts on the set G/H of right cosets by right multiplication. Clearly, the action of G on x^G "is the same", as the action of G on G/G_x by $(G_xg)h = G_x(gh)$. It is also clear, that H_G is the kernel of the action of G on G/H.

If A is a group and $B \leqslant \mathrm{Sym}_n$, then by $A \wr B$ we denote the permutational wreath product, i.e. $A\wr B = (A_1 \times \ldots \times A_n) \rtimes B$, where B permutes A_i-s and $A_i \simeq A$ for every i.

In view of its importance for our purposes, we discuss the permutation wreath product in more details. If $(a_1, \ldots, a_n) \in A_1 \times \ldots \times A_n$ and $b \in B \leqslant \mathrm{Sym}_n$, then

$$b : (a_1, \ldots, a_n) \mapsto (a_{(1)b^{-1}}, \ldots, a_{(n)b^{-1}})$$

determines an action of B on $A_1 \times \ldots \times A_n$ ($A_i \simeq A$) and so determines the semidirect product $(A_1 \times \ldots \times A_n) \rtimes B = A \wr B$.

Recall that $A_1 \times \ldots \times A_n$ can be defined as the set

$$\{f : \{1, \ldots, n\} \to \cup_{i=1}^n A_i \mid f(i) \in A_i\},$$

with multiplication $(f \cdot g)(i) = f(i) \cdot g(i)$. Given A and $B \leqslant \mathrm{Sym}_n$ the identity

$$A \wr B = \{(f, b) \mid f \in A_1 \times \ldots \times A_n; b \in B\}$$

holds. Moreover $f^b(i) = (b^{-1} \cdot f \cdot b)(i) = f((i)b^{-1})$.

A normal series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_k = 1$$

is called a chief series, if G_{i-1}/G_i is a minimal normal subgroup of G/G_i for $i = 1, \ldots, k-1$. A composition series

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_n = 1$$

is called an (rc)-series, if it is a refinement of a chief series of G.

Theorem 0. (Jordan-Hölder Theorem)
Let

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_n = 1$$

be a composition series of G and denote the quotient G_{i-1}/G_i by S_i. Then for every composition series

$$G = H_0 \trianglerighteq H_1 \trianglerighteq \ldots \trianglerighteq H_m = 1$$

$m = n$ and for every $T_i = H_{i-1}/H_i$ there exists S_j such that $T_i \simeq S_j$.

Let A, B, H be subgroups of G such that $B \trianglelefteq A$. Then $N_H(A/B) := N_H(A) \cap N_H(B)$ is the normalizer of A/B in H. If $x \in N_H(A/B)$, then x induces an automorphism of A/B by $Ba \mapsto Bx^{-1}ax$. Thus there exists a homomorphism $N_H(A/B) \to \mathrm{Aut}(A/B)$. The image of $N_H(A/B)$ under this homomorphism is denoted by $\mathrm{Aut}_H(A/B)$ and is called a group of H-induced automorphisms of A/B. If $B = 1$, then we write just $\mathrm{Aut}_H(A)$.

Usually we use the notion of H-induced subgroups in the case, when A/B is a composition factor of G so that A, B are members of a composition series. In such case the group of induced automorphisms is not determined by the composition factor of G only, but also depends on the choice of composition series (see Example 2 below). Notice also that for any $A, B, H \leqslant G$ such that $B \trianglelefteq A$, we have $N_H(A/B) \leqslant N_G(A/B)$ and $C_H(A/B) = C_G(A/B) \cap N_H(A/B)$, so natural isomorphism $N_H(A/B)/C_H(A/B) \to (N_H(A/B)C_G(A/B))/C_G(A/B)$ gives us a natural embedding of $\mathrm{Aut}_H(A/B)$ into $\mathrm{Aut}_G(A/B)$, and we always consider $\mathrm{Aut}_H(A/B)$ as a subgroup of $\mathrm{Aut}_G(A/B)$.

The next example shows that groups of induced automorphisms provide additional information to the information that can be derived from a composition series of G.

Example 1.

Groups $\mathbb{Z}_6$ and Sym_3 have the same composition series: $\mathrm{Sym}_3 \geqslant \mathbb{Z}_3 \geqslant 1$ and $\mathbb{Z}_6 \geqslant \mathbb{Z}_3 \geqslant 1$. However, $\mathrm{Aut}_{\mathrm{Sym}_3}(\mathbb{Z}_3) \simeq \mathbb{Z}_2$, while $\mathrm{Aut}_{\mathbb{Z}_6}(\mathbb{Z}_3) = 1$, so we could discern these groups, if we would know not only composition series, but also groups of induced automorphisms.

Example 2 (Groups of induced automorphisms do depend on a composition series)
Let G be a finite simple group, possessing an outer automorphism τ of order 2 (for example, $G \simeq \mathrm{Alt}_n$ and $\langle G, \tau\rangle = \mathrm{Sym}_n$). Consider an elementary abelian group $\langle x, y\rangle$ of order 4. Then $\langle G, \tau\rangle$ acts on $\langle x, y\rangle$ by $x^\tau \mapsto x$, $y^\tau \mapsto x \cdot y$, and G is the kernel of the action. Consider $L = \langle x, y\rangle \rtimes \langle G, \tau\rangle$. Then L has two composition series:

$$1 \leqslant \langle x\rangle \leqslant \langle x, y\rangle \leqslant \langle x, y\rangle \times G \leqslant L$$

$$1 \leqslant \langle y\rangle \leqslant \langle y\rangle \times G \leqslant \langle x, y\rangle \times G \leqslant L.$$

Clearly L has a unique nonabelian composition factor isomorphic to G (and we also denote it by G), but in the first series we have $\mathrm{Aut}_L(G) = \mathrm{Aut}_L((\langle x, y\rangle \times G)/\langle x, y\rangle) = G \rtimes \langle\tau\rangle$, while in the second series $\mathrm{Aut}_L(G) = \mathrm{Aut}_L((\langle y\rangle \times G)/\langle y\rangle) = G$.

Theorem 1. (Generalized Jordan-Hölder Theorem)
Let

$$G = G_0 \trianglerighteq G_1 \trianglerighteq \ldots \trianglerighteq G_n = 1$$

be an (rc)-series of G, denote the quotient G_{i-1}/G_i by S_i. Assume that

$$G = H_0 \trianglerighteq H_1 \trianglerighteq \ldots \trianglerighteq H_n = 1$$

is a composition series of G and $T_i = H_{i-1}/H_i$. Then, for every T_i there exists S_j such that $T_i \simeq S_j$ and $\mathrm{Aut}_G(T_i) \leqslant \mathrm{Aut}_G(S_j)$. Moreover, if the second series is an (rc)-series, then $\mathrm{Aut}_G(T_i) = \mathrm{Aut}_G(S_j)$.

The theorem follows from the following lemma by evident induction.

Lemma 1.
Let H be a normal subgroup of G, $S = (A/H)/(B/H)$ be a composition factor of G/H and L be a subgroup of G. Then $\mathrm{Aut}_L(A/B) \simeq \mathrm{Aut}_{LH/H}((A/H)/(B/H))$.

Proposition 1. (Huppert, Endliche gruppen, Hauptsatz 1.4, p 413.)
Let H be a subgroup of index n of G, consider $G = \cup_{i=1}^n Hr_i$, the union of right cosets with representatives r_i-s. For every $g \in G$ define a permutation $\mathbf{P}(g)$ of $\{1, \ldots, n\}$ and $h_i(g) \in H$ $(i = 1, \ldots, n)$ by

$$r_i g = h_i(g) r_{(i)\mathbf{P}(g)}.$$

Thus $\mathbf{P}(G)$ is a subgroup of Sym_n.

Then the mapping $\mathbf{M}$, given by $g \mapsto \mathbf{M}(g) = (f, \mathbf{P}(g))$, where $f(i) = h_i(g)$, is an injective embedding of G into $H \wr \mathbf{P}(G)$. The mapping $\mathbf{M}$ is called the monomial representation of G over H.

Proof.

Equalities

$$h_i(gg')r_{(i)\mathbf{P}(gg')} = r_i(gg') = (r_i g)g' = h_i(g)r_{(i)\mathbf{P}(g)}g' = h_i(g)h_{(i)\mathbf{P}(g)}(g')r_{((i)\mathbf{P}(g))\mathbf{P}(g')}$$

imply $\mathbf{P}(gg') = \mathbf{P}(g)\mathbf{P}(g')$ and $h_i(gg') = h_i(g)h_{(i)\mathbf{P}(g)}(g')$. Thus $\mathbf{M}(gg') = (f, \mathbf{P}(g)\mathbf{P}(g'))$, where $f(i) = h_i(gg') = h_i(g)h_{(i)\mathbf{P}(g)}(g')$. Since

$$\mathbf{M}(g)\mathbf{M}(g') = (f_g, \mathbf{P}(g)) \cdot (f_{g'}, \mathbf{P}(g')) = (f_g \cdot (\mathbf{P}(g)f_{g'}\mathbf{P}(g)^{-1}), \mathbf{P}(g)\mathbf{P}(g')),$$

where $f_g(i) = h_i(g)$ and $(\mathbf{P}(g)f_{g'}\mathbf{P}(g)^{-1})(i) = h_{(i)\mathbf{P}(g)}(g')$, we derive the equality $\mathbf{M}(gg') = \mathbf{M}(g)\mathbf{M}(g')$.

Now from $\mathbf{M}(g) = 1$ we obtain that $r_i g = r_i$ for each i, hence $g = 1$. □

Inductive arguments often reduces problems to the consideration of the following configuration:

G possesses a unique minimal normal subgroup $T = L_1 \times \ldots \times L_k$, where $L_1, \ldots, L_k$ are nonabelian simple groups. Our next goal is to derive as complete information on the structure of G as possible, and find a canonical form of an element from G.

Since all L_i-s are nonabelian, and T is a unique minimal normal subgroup, we obtain that $C_G(T) = 1$ and G acts by conjugation on $\{L_1, \ldots, L_k\}$. Now the action of G by conjugation on $\{L_1, \ldots, L_k\}$ is permutationally equivalent to the action of G on right cosets by $N_G(L_1)$. Denote the image of G in Sym_k by P. Proposition 1 implies that there exists an injective embedding $\mathbf{M} : G \to N_G(L_1) \wr P$.

All normalizers $N_G(L_i)$-s are evidently conjugate in G, so we can identify $N_G(L_1) \wr P$ in a natural way:

$$N_G(L_1) \wr P \simeq (N_G(L_1) \times \ldots \times N_G(L_k)) \rtimes P$$

and assume that $\mathbf{M}$ send G into $(N_G(L_1) \times \ldots \times N_G(L_k)) \rtimes P$. Now $C_G(L_1) \times \ldots \times C_G(L_k)$ is a normal subgroup of $N_G(L_1) \wr P$ and

$$((N_G(L_1) \times \ldots \times N_G(L_k)) \rtimes P) / (C_G(L_1) \times \ldots \times C_G(L_k)) \simeq \mathrm{Aut}_G(L_1) \wr P,$$

hence there exists a homomorphism $\varphi : G \to \mathrm{Aut}_G(L_1) \wr P$. Moreover, since $C_G(L_1) \times \ldots \times C_G(L_k) = C_{N_G(L_1) \wr P}(T)$, it follows that $\mathrm{Ker}(\varphi) \leqslant C_G(T) = 1$, in particular, φ is injective.

Thus the following theorem is obtained

Theorem 2.
Assume that G possesses a unique minimal normal subgroup $T = L_1 \times \ldots \times L_k$, where $L_1, \ldots, L_k$ are (isomorphic) nonabelian simple groups. In particular, G acts by conjugation on $\{L_1, \ldots, L_k\}$. Denote the image of G in Sym_k by P; the factor $N_G(L_1)/C_G(L_1)$ by definition is denoted $\mathrm{Aut}_G(L_1)$. Then there exists an injective embedding $\varphi : G \to \mathrm{Aut}_G(L_1) \wr P$.

Notice that the embedding $G \to \mathrm{Aut}(L_1) \wr \mathrm{Sym}_k$ is known and can be found in any textbook on group theory. However, the embedding from Theorem 2 is more precise and is closed to isomorphism. Moreover, the exact construction of the embedding is also important.

Lecture 2. Subgroups of simple groups.

Here and below the term "simple group" always means "nonabelian simple group". A group G is said to be almost simple if there exists a simple group S with

$$S \simeq \mathrm{Inn}(S) \leqslant G \leqslant \mathrm{Aut}(S).$$

Classification Theorem.
If G is a finite simple group, then G is isomorphic to

1. alternating group of degree $n \geqslant 5$;
2. simple group of Lie type;
3. sporadic group.

Recall that $G \leqslant \mathrm{Sym}_n$ is called transitive, if for every $1 \leqslant i < j \leqslant n$ there exists $g \in G$ with $(i)g = j$. Recall also that $G \leqslant \mathrm{Sym}_n$ is called primitive if there does not exist a nontrivial partition $\{1, \ldots, n\} = X_1 \sqcup \ldots \sqcup X_k$ such that for every $1 \leqslant i \leqslant k$ and for every $g \in G$ there exists $j \in \{1, \ldots, k\}$ with $(X_i)g \subseteq X_j$. It is clear from the definition that a primitive group is always transitive.

Now, if $G \leqslant Sym_n$ is not transitive, then $G \leqslant \mathrm{Sym}_{n_1} \times \mathrm{Sym}_{n_2}$, where $n_1 + n_2 = n$. If $G \leqslant \mathrm{Sym}_n$ is transitive, but is not primitive, then $G \leqslant \mathrm{Sym}_k \wr \mathrm{Sym}_m$, where $k \cdot m = n$. Finally, if G is primitive, then its socle N is transitive and N is a product of isomorphic simple groups. If N is abelian, then $N \simeq \mathbb{Z}_p^k$, $n = p^k$, and $G \leqslant N \rtimes \mathrm{GL}_k(p)$. Moreover, the image of G in $\mathrm{GL}_k(p)$ under the natural homomorphism is an irreducible subgroup. If N is nonabelian, then either G is almost simple, or we can construct for $G/C_N(G)$ the embedding from Theorem 2 of the previous lecture.

Assume that $\mathbb{F}$ is an algebraically closed field. For every $I \subseteq \mathbb{F}[T_1, \ldots, T_n]$ denote by $id(I)$ the ideal generated by I and by $\sqrt{id(I)}$ the radical ideal generated by I. Define

$$\mathcal{V}(I) = \{(x_1, \ldots, x_n) \in \mathbb{F}^n \mid \forall f \in I, f(x_1, \ldots, x_n) = 0\}.$$

For every $X \subseteq \mathbb{F}^n$ define

$$\mathcal{I}(X) = \{f \in \mathbb{F}[T_1, \ldots, T_n] \mid \forall (x_1, \ldots, x_n) \in X, f(x_1, \ldots, x_n) = 0\}.$$

Clearly $\mathcal{V}(I) = \mathcal{V}(id(I)) = \mathcal{V}(\sqrt{id(I)})$.

Lemma 1.
For every ideals I, J of $\mathbb{F}[T_1, \ldots, T_n]$ and for every set of ideals $\{I_\alpha \mid \alpha \in A\}$ of $\mathbb{F}[T_1, \ldots, T_n]$ the equalities

$$\mathcal{V}(I) \cup \mathcal{V}(J) = \mathcal{V}(I \cap J) \text{ and } \bigcap_{\alpha \in A} \mathcal{V}(I_\alpha) = \mathcal{V}\left(\sum_{\alpha \in A} I_\alpha\right)$$

hold.

Hilbert Nullstellensatz
For every $I \subseteq \mathbb{F}[T_1, \ldots, T_n]$ the equality $\sqrt{id(I)} = \mathcal{I}(\mathcal{V}(I))$ holds.

If $X \subseteq \mathbb{F}^n$ can be written as $\mathcal{V}(I)$ for appropriate $I \subseteq \mathbb{F}[T_1, \ldots, T_n]$, then X is called an affine variety. Lemma 1 and Hilbert Nullstellensatz imply that the set of all affine varieties of $\mathbb{F}^n$ contains $\varnothing, \mathbb{F}^n$ and is closed under finite unions and intersections. Therefore it is possible to determine a topology (called the Zariski topology) on $\mathbb{F}^n$ (and so make $\mathbb{F}^n$ to be a topological space) setting affine varieties to be closed subsets. For every affine variety X define $\mathbb{F}[X] = \mathbb{F}[T_1, \ldots, T_n]/\mathcal{I}(X)$.

Recall that a topological space is called irreducible, if it is not a union of two proper closed subsets, and a topological space is called connected, if it is not a disjoint union of two proper closed subsets. It is not difficult to check that an affine variety is irreducible if and only if $\mathbb{F}[X]$ is an integral domain. For irreducible X denote $\dim(X)$ as the transcendent degree of the field of fractions $\mathbb{F}(X)$ over $\mathbb{F}$.

Hilbert Basissatz
If R is a Noetherian ring, then $R[T]$ is also Noetherian. In particular, each ideal of $\mathbb{F}[T_1, \ldots, T_n]$ is finitely generated.

Hilbert Basissatz implies that every affine variety X is a Noetherian topological space, in particular, it can be uniquely written as $X = X_1 \cup \ldots \cup X_n$, where each X_i is a maximal irreducible closed subset of X. $X_1, \ldots, X_n$ are called irreducible components of X. By definition, $\dim(X) = \max_i \dim(X_i)$.

If U is an open subset of $\mathbb{F}^n$, then $X = \mathbb{F}^n \setminus U$ is closed and $\mathcal{I}(X)$ is generated by $f_1, \ldots, f_k \in \mathbb{F}[T_1, \ldots, T_k]$. Define $\mathbb{F}[U]$ to be the subring of $\mathbb{F}(T_1, \ldots, T_n)$ generated by $\mathbb{F}[T_1, \ldots, T_n]$ and $1/f_1, \ldots, 1/f_k$. Now induced topology on U can be defined as a Zariski topology with $\mathbb{F}[U]$ instead of $\mathbb{F}[T_1, \ldots, T_n]$. In particular, $\mathrm{GL}_n(\mathbb{F})$ is an open subset of $\mathbb{F}^{n^2} = \mathrm{Mat}_n(\mathbb{F})$ and $\mathbb{F}[\mathrm{GL}_n(\mathbb{F})]$ is generated by $T_{i,j}$, $1 \leqslant i, j \leqslant n$ and $1/\det(T_{i,j})$. Closed subgroups of $\mathrm{GL}_n(\mathbb{F})$ are called linear algebraic groups.

Let G be a linear algebraic group. Denote by G^0 the minimal closed subgroup of finite index of G, the so-called connected component of G. G^0 is a normal subgroup of G and G^0 is an irreducible component. The remaining irreducible components of G are the cosets G/G^0, in particular, G is a disjoint intersection of its irreducible component and G is an irreducible variety if and only if G is a connected variety. Since G is linear, the term "irreducible" is used in contest of linear groups, so if G is an irreducible variety, then G is called connected.

Lemma 2.
Let $\{X_i \mid i \in I\}$ be a set of irreducible closed subsets of a linear algebraic group G. Assume also that $e \in X_i$ for all $i \in I$. Then $\langle X_i \mid i \in I\rangle$ is a closed connected subgroup of G.

Now we assume that $\mathbb{F} = \overline{F}_p$ is the algebraic closure of a finite field of characteristic p, in particular, each element of $\mathrm{GL}_n(\mathbb{F})$ has finite order. $g \in G$ is called semisimple, g is conjugate to a diagonal matrix or, equivalently, if $(|g|, p) = 1$. $g \in G$ is called unipotent, if $(g - e)^n = 0$ or, equivalently, if $|g| = p^k$ for some $k \geqslant 1$.

Lemma 3.
Let $G \leqslant \mathrm{GL}_n(\mathbb{F})$ be a linear algebraic group. Then for every $g \in G$ there exist the unique $g_s \in G$ and $g_u \in G$ such that g_s is semisimple, g_u is unipotent and $g = g_s \cdot g_u$. The expression $g = g_s \cdot g_u$ is called the Jordan decomposition of g.

Notice that if $\mathbb{F} = \overline{F}_p$, then Lemma 3 holds for every subgroup of $\mathrm{GL}_n(\mathbb{F})$, while in general case we need to assume that G is closed. A subgroup H of G is called unipotent, if it consists of unipotent elements.

G is called torus, if G is isomorphic (as an algebraic group) to $\mathbb{F}^\times \times \ldots \times \mathbb{F}^\times$, where $\mathbb{F}^\times$ is the multiplicative group of $\mathbb{F}$. Maximal connected closed solvable subgroup of G is called a Borel subgroup. Every closed subgroup P of G with $B \leqslant P$ is called parabolic. The maximal closed connected normal unipotent subgroup of G is called the unipotent radical and is denoted by $R_u(G)$. The maximal closed connected normal solvable subgroup of G is called the radical and is denoted by $R(G)$.

Lemma 4.
Let G be a connected linear algebraic group.

1. If G is solvable, then $G = R_u(G) \rtimes T$, where T is a maximal torus of G.
2. If B is a Borel subgroup of G, $U = R_u(B)$ and T is a maximal torus of B, then U is a maximal closed connected unipotent subgroup of G and T is a maximal torus of G. Moreover, all Borel subgroups, all maximal closed connected unipotent subgroups and all maximal tori are conjugate in G and $\cup_{g\in G} B^g = G$, $\cup_{g\in G} U^g$ is the set of all unipotent elements of G, and $\cup_{g\in G} T^g$ is the set of all semisimple elements of G. The dimension of a maximal torus is called rank of G.
3. $N_G(B) = B$, in particular, B is a maximal solvable subgroup of G, every parabolic subgroup P of G is connected and $N_G(P) = P$.

4. If S is a torus of G and B is a Borel subgroup containing S, then $C_G(S)$ is connected subgroup of maximal rank of G and $B \cap C_G(S)$ is a Borel subgroup of $C_G(S)$.

Linear algebraic group G is called reductive, if $R_u(G)$ is trivial and is called semisimple, if $R(G)$ is trivial. G is called simple if it does not possess proper normal subgroups of positive dimension. For every connected reductive algebraic group G it is possible to define the root system $\Phi = \Phi(G)$. A connected simple linear algebraic group is uniquely determined by its root system Φ (which must be irreducible), field $\mathbb{F}$, and the structure of (finite) center $Z(G)$. If $Z(G) = 1$, then G is called adjoint, and if $Z(G)$ is the largest possible, then G is called simply connected.

Lemma 5.
Let G be a connected reductive linear algebraic group.

1. $R(G) = Z(G)^0 = Z$, $G = [G,G] \circ Z$, where $[G,G]$ is connected and semisimple.
2. For every semisimple $s \in G$, $C_G(s)^0$ is reductive and every unipotent element from $C_G(s)$ lies in $C_G(s)^0$.
3. If G is semisimple, then $G = G_1 \circ \ldots \circ G_m$, where all G_i-s are simple and $\Phi(G) = \Phi(G_1) \sqcup \ldots \sqcup \Phi(G_m)$.

Let H be a closed subgroup of a connected reductive linear algebraic group G such that $R_u(H)$ is nontrivial. Set $H_0 = H$ and $U_0 = R_u(H)$. For $i \geqslant 1$ define $H_i = N_G(U_{i-1})$ and $U_i = R_u(H_i)$. Clearly $H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$ and $U_0 \subseteq U_1 \subseteq U_2 \subseteq \ldots$

Lemma 6. (Borel-Tits Theorem)
In the above notation the series $H_0 \subseteq H_1 \subseteq H_2 \subseteq \ldots$ is stabilized after a finite number of steps, and $\cup_{i\geqslant 0} H_i = P$ is a parabolic subgroup of G. Every parabolic subgroup P of G has a decomposition $P = U \rtimes L$, where U is the unipotent radical of P and L is a reductive subgroup of maximal rank of G. Moreover, $\Phi(L)$ is a subsystem (more precisely additively closed subsystem) of $\Phi(G)$.

Recall that irreducible root systems consist of four infinite series: $A_n, n \geqslant 1$, $B_n, n \geqslant 2$, $C_n, n \geqslant 3$, $D_n, n \geqslant 4$ and five exceptional: E_6, E_7, E_8, F_4, G_2. Every irreducible root system is uniquely determined by its Dynkin diagram or by its Cartan matrix. If $\Delta(\Phi)$ is the determinant of the Cartan matrix of Φ, then $\Delta(A_n) = n+1$, $\Delta(D_n) = 4$, $\Delta(B_n) = \Delta(C_n) = \Delta(E_7) = 2$, $\Delta(E_6) = 3$, and $\Delta(E_8) = \Delta(F_4) = \Delta(G_2) = 1$.

Extended Dynkin diagram can be obtain form the Dynkin diagram by the following procedure. We add the vertex $-r_0$, where r_0 is the root of the highest weight, and connect it by the usual rule. Now there exists an algorithm due to Borel and De Siebental and, independently Dynkin, which allows to obtain all additively closed subsystem of Φ. One should consider the extended Dynkin diagram, remove some vertices and repeat the procedure for remaining connected component. Diagrams obtaining in this way are precisely Dynkin diagrams of additively closed subsystems of Φ.

Lemma 7.
Let G be a simple connected algebraic group with root system Φ and s be a semisimple element of G.

Then $C_G(s)^0$ is a reductive subgroup of maximal rank of G, $\Phi(C_G(s)^0)$ is an additively closed subsystem of Φ, and $|C_G(s)/C_G(s)^0|$ divides $\Delta(\Phi)$.

Consider the automorphism $\varphi : \mathrm{GL}_n(\overline{F}_p) \to \mathrm{GL}_n(\overline{F}_p)$ acting by $\varphi : (a_{i,j}) \mapsto (a_{i,j}^p)$, we call φ a canonical Frobenius map. If $\overline{G} \leqslant \mathrm{GL}_n(\overline{F}_p)$, then a surjective endomorphism $\sigma : \overline{G} \to \overline{G}$ is called a Steinberg map or a Frobenius map, if the set $\overline{G}_\sigma$ of σ-stable points is finite. It is known, that there exist $\alpha, k \geqslant 1$ such that up to conjugation in $\mathrm{GL}_n(\overline{F}_p)$ we have $\varphi^\alpha|_{\overline{G}} = \sigma^k|_{\overline{G}}$. Any group, satisfying

$$O^{p'}(\overline{G}_\sigma) \leqslant G \leqslant \overline{G}_\sigma$$

is called a finite group of Lie type. Usually (but not always!!!) $\mathbb{F}_{p^\alpha}$ is called the definition field for G, while $\mathbb{F}_{p^{\alpha/k}}$ is called the base field for G.

If $\overline{H} \leqslant \overline{G}$ is σ-invariant and $\overline{H}$ is a Borel subgroup, a parabolic subgroup, a reductive subgroup of maximal rank, a maximal torus, a normalizer of maximal torus, then $\overline{H} \cap G$ is called a Borel subgroup, a parabolic subgroup, a reductive subgroup of maximal rank, a maximal torus, a normalizer of maximal torus of G, respectively.

Lang-Steinberg Theorem
Let $\overline{G}$ be a connected simple algebraic group over $\overline{F}_p$ and $\sigma : \overline{G} \to \overline{G}$ is a Steinberg map. Then the map $x \mapsto x^{-1}x^\sigma$ is surjective.

This theorem together with the precise form of σ allows to transfer results about subgroups of linear algebraic groups to finite groups of Lie type. In particular, the theorem allows to study subgroups, having either a normal p-subgroup, or a central semisimple element.

Lecture 3. Carter subgroups.

Recall that a nilpotent selfnormalizing subgroup is called a Carter subgroup.

Theorem 3. (R.Carter, 1962)
Let G be a finite solvable group. Then G possesses a Carter subgroup and all Carter subgroups of G are conjugate.

Problem 1.
Are Carter subgroups conjugate?

Problem 2.
When given group possesses a Carter subgroup?

Example 1.
1. Consider $G = \mathrm{Sym}_3$, $H = \mathrm{Alt}_3$, so that $H \trianglelefteqslant G$. Then a Carter subgroup K of G has order 2, while H is a Carter subgroup of H, in particular, $K \cap H = 1$.

2. Consider $G = \mathrm{Sym}_4$, $H = \mathrm{Alt}_4$, so that $H \trianglelefteqslant G$. Then a Carter subgroup K of G has order 8, while a Carter subgroup F of H has order 3, in particular, $K \cap F = 1$, however $K \cap H \neq 1$.

3. Consider $G = \mathrm{Sym}_5$, $H = \mathrm{Alt}_5$, again $H \trianglelefteqslant G$. A Carter subgroup K of G has order 8, while H does not possess a Carter subgroups, however $K \cap H \neq 1$.

4. Consider $G = \mathrm{Sym}_6$, $H = \mathrm{Alt}_6$, again $H \trianglelefteqslant G$. If K is a Carter subgroup of G, then K is a Sylow 2-subgroup of G and $K \cap H$ is a Carter subgroup of H.

Lemma 1.
Let G be a finite group possessing a Carter subgroup K. Assume that there exists $B = T_1 \times \ldots \times T_k \trianglelefteqslant G$ such that $T_1 \simeq \ldots \simeq T_k \simeq T$, G acts by conjugation on the set $\{T_1, \ldots, T_k\}$, and $Z(T) = 1$. Suppose also that $G = KB$. Then $\mathrm{Aut}_K(T_i)$ is a Carter subgroup of $\mathrm{Aut}_G(T_i)$ for $i = 1, \ldots, k$.

Proof. We proceed by induction on k. Clearly we may assume that G acts transitively by conjugation on $\{T_1, \ldots, T_k\}$. Let $\Delta_1, \ldots, \Delta_p$ a system of imprimitivity for the action such that G acts on the system primitively. Since B lies in the kernel of the action and G/B is nilpotent it follows that p is a prime. Set

$$S_i = \prod_{T_j \in \Delta_i} T_j.$$

Clearly, $S_1 \simeq S_2 \simeq \ldots \simeq S_p \simeq S$ and $Z(S) = 1$. Now $|K : N_K(S_1)| = p$, so $N_K(S_1)$ is a normal subgroup of prime index k of K and $N_K(S_1)$ normalizes S_i for $i = 1, \ldots, k$. Evident computations using embedding from Theorem 2 and inductive argumets complete the proof. □

Theorem 4. (F.Dalla Volta and M.C.Tamburini (1998); EV (2006))
Let $G = G_0 \trianglerighteqslant G_1 \trianglerighteqslant \ldots \trianglerighteqslant G_n = 1$ be an (rc)-series of G. Assume also that Carter subgroups in every L such that $G_{i-1}/G_i \leqslant L \leqslant \mathrm{Aut}_G(G_{i-1}/G_i)$ are conjugate for each $i = 1, \ldots, n$ (however they may fail to exist). Then Carter subgroups of G are conjugate.

Proof. We prove the theorem by induction on $|G|$. Let K and H be nonconjugate Carter subgroups of G. Theorem 1 implies that for every normal subgroup N of G the factor group G/N satisfies to the condition of the theorem, in particular, we obtain that G possesses a unique minimal normal subgroup $B = T_1 \times \ldots \times T_k$, where all T_i-s are isomorphic simple groups. Moreover $G = HB = KB$. Consider the embedding $\varphi : G \to \mathrm{Aut}_G(T_1) \wr P$ defined in Theorem 2. We have that images of both H and K in Sym_k equal P and, by Lemma 1, both $\mathrm{Aut}_H(T_1)$ and $\mathrm{Aut}_K(T_1)$ are Carter subgroups of $\mathrm{Aut}_G(T_1)$. The conditions of the theorem implies that $\mathrm{Aut}_H(T_1)$ and $\mathrm{Aut}_K(T_1)$ are conjugate in $\mathrm{Aut}_G(T_1)$, so we may assume that $\mathrm{Aut}_H(T_1) = \mathrm{Aut}_K(T_1)$. Hence both H^φ and K^φ are Carter subgroups of a solvable group $G^\varphi \cap \mathrm{Aut}_K(T_1) \wr P$, therefore they are conjugate. □

We say that a finite group G satisfies condition **(C)**, if for an (rc)-series $G = G_0 \trianglerighteqslant G_1 \trianglerighteqslant \ldots \trianglerighteqslant G_n = 1$ of G, Carter subgroups in every L such that $G_{i-1}/G_i \leqslant L \leqslant \mathrm{Aut}_G(G_{i-1}/G_i)$ are conjugate for each $i = 1, \ldots, n$.

Lemma 2.
Assume that G is a finite group. Let K be a Carter subgroup of G, with center $Z(K)$. Assume also that $e \neq z \in Z(K)$ and $C_G(z)$ satisfies **(C)**.

(1) Every subgroup Y which contains K and satisfies **(C)** is self-normalizing in G.

(2) No conjugate of z in G, except z, lies in $Z(K)$.

(3) If H is a Carter subgroup of G, non-conjugate to K, then z is not conjugate to any element in the center of H.

In particular the centralizer $C_G(z)$ is self-normalizing in G, and z is not conjugate to any power $z^k \neq z$.

Proof. Since Carter subgroups of Y are conjugate and $Y \trianglelefteq N_G(Y)$, Frattini argument implies the first statement. (2) and (3) follows from (1). ☐

Lemma 3.
If $n \geqslant 5$, then for every element $g \in \mathrm{Sym}_n$ of odd prime order p there exists k and $x \in \mathrm{Alt}_n$ such that $g^x = g^k \neq g$.

Proof. Clearly there exists $x \in \mathrm{Sym}_n$ such that $g^x = g^{-1}$. If the decomposition of x into the product of independent cycles consists of $m > 1$ cycles, then $C_{\mathrm{Sym}_n}(g) \geqslant \mathbb{Z}_p \wr \mathrm{Sym}_m$, and so there exist $y \in C_{\mathrm{Sym}_n}(g) \setminus \mathrm{Alt}_n$, hence either $x \in \mathrm{Alt}_n$ or $xy \in \mathrm{Alt}_n$. Now assume that g is a cycle of length p. If $p = 3$, then $n - p \geqslant 2$, so there exists a transposition t centralizing g and again either $x \in \mathrm{Alt}_n$ or $xt \in \mathrm{Alt}_n$. If $p \geqslant 5$, then $C_{\mathrm{Sym}_p}(g) = \mathbb{Z}_p$, while $N_{\mathrm{Sym}_p}(g) = \mathbb{Z}_p \rtimes \mathbb{Z}_{p-1}$. So there exists $y \in \mathrm{Alt}_p \leqslant \mathrm{Alt}_n$ centralizing but not normalizing g and the lemma follows. ☐

Corollary
If $n \geqslant 5$ and K is a Carter subgroup of Sym_n or Alt_n, then K is a Sylow 2-subgroup. In particular, Carter subgroups in an almost simple group with simple socle isomorphic to Alt_n are conjugate.

In the following finite simple groups each element of odd prime order is conjugate to its nontrivial power:
alternating, sporadic; $A_1(p^t)$, $B_n(p^t)$; $C_n(p^t)$, t is even if $p = 3$;${}^2B_2(2^{2n+1})$,$G_2(p^t)$, $F_4(p^t)$, ${}^2F_4(2^{2n+1})$ $E_7(p^t)$, $p \neq 3$; $E_8(p^t)$, $p \neq 3, 5$ ${}^3D_4(p^t)$; $D_{2n}(p^t)$, ${}^2D_{2n}(p^t)$, where t is even if $p = 3$, ${}^2G_2(3^{2n+1})$.
If a finite simple group lies in this list, then its Carter subgroup (if it exists) should coincide with a Sylow 2-subgroup, in particular, they are conjugate.

Lemma 4.
Let $\overline{G}$ be a connected simple linear algebraic group over a field of characteristic p, σ be a Steinberg map of $\overline{G}$ and $G = \overline{G}_\sigma$ be a finite group of Lie type. Let φ be a field or a graph-field automorphism of G (if φ is graph-field, then corresponding graph automorphism has order 2) and let φ' be an element of $(G \leftthreetimes \langle\varphi\rangle) \setminus G$ such that $|\varphi'| = |\varphi|$ and $G \leftthreetimes \langle\varphi\rangle = G \leftthreetimes \langle\varphi'\rangle$.
Then there exists $g \in G$ such that $\langle\varphi\rangle^g = \langle\varphi'\rangle$. In particular, if $G/O^{p'}(G)$ is a 2-group and φ is of odd order, then g can be chosen in $O^{p'}(G)$.

Lemma 5.
Let K be a Carter and N be a normal subgroups of a finite group G. Assume that either KN satisfies **(C)** (this condition holds, in particular, if either G satisfies **(C)** or N is solvable) or $KN = G$.
Then KN/N is a Carter subgroup of G/N.

Proof. Clearly we may assume that $KN \neq G$. Evidently KN/N is nilpotent, so we remain to prove that KN/N is selfnormalizing or, equivalently, that KN is selfnormalizing. If x normalizes KN, then K^x is a Carter subgroup of KN. Now KN satisfies **(C)**, so there exists $y \in KN$ with $K^x = K^y$. So $xy^{-1} \in N_G(K) = K \leqslant KN$, therefore $x \in KN$. ☐
A finite group G is said to satisfy **(ESyl2)**, if for its Sylow 2-subgroup Q the equality $N_G(Q) = QC_G(Q)$ holds. In other words, G satisfies **(ESyl2)**, if every element of odd order, normalizing a Sylow 2-subgroup Q of G, centralizes Q.

Lemma 6.
G contains a Carter subgroup K with $Syl_2(K) \subseteq Syl_2(G)$ if and only if G satisfies **(ESyl2)**.

Proof. Assume that G contains a Carter subgroup K such that $Q \leqslant K$ for some $Q \in Syl_2(G)$. K is nilpotent, hence Q is normal in K and $K \leqslant QC_G(Q) \trianglelefteq N_G(Q)$. By Feit-Thompson theorem, we obtain that $N_G(Q)$ is solvable. Lemma 2(1) implies that $QC_G(Q)$ is self-normalizing in $N_G(Q)$, so $N_G(Q) = QC_G(Q)$.
Assume now that $N_G(Q) = QC_G(Q)$ for some $Q \in Syl_2(G)$, i. e., the equality $N_G(Q) = Q \times O(C_G(Q))$ holds. Since $O(C_G(S))$ is of odd order, it is solvable. Therefore it contains a Carter subgroup K_1. Consider a nilpotent subgroup $K = Q \times K_1$ of G. Assume that $x \in N_G(K)$, then $x \in N_G(Q)$. But K is a Carter subgroup of $N_G(Q)$, hence $x \in K$ and K is a Carter subgroup of G. ☐

Lemma 7.
Assume that G possesses normal subgroups $G_1, \ldots, G_k$ of G such that $G_1 \cap \ldots \cap G_k \cap Q \leqslant Z(N_G(Q))$ for $Q \in Syl_2(G)$. Assume that G/G_i satisfies **(ESyl2)** for all i. Then G satisfies **(ESyl2)**. .

Now the general line for the proof that Carter subgroups in a finite almost simple group G are conjugate is the following. First we reduce to the case, when $|G : Soc(G)| = 2^t$. Assume that K is a Carter subgroup of G and $K \notin Syl_2(G)$. Then $K \cap Soc(G)$ contains a central element x of odd prime order. By induction we may assume, that $C_G(x)$ satisfies **(C)**, in particular, its Carter subgroups are conjugate and "contain" Sylow 2-subgroup of composition factors. In particular we can find normal subgroups of $C_G(x)$ satisfying conditions of Lemma 7. So, $C_G(x)$ satisfies **(ESyl2)**, in particular K contains a Sylow 2-subgroup of $C_G(x)$. If x is unipotent, then we obtain that its centralizer contains a "large" 2-subgroup, whose centralizer does not contains unipotent elements. If x is semisimple, then we repeat the arguments for some central involution of $K \cap Soc(G)$ and obtain that $K \cap Soc(G)$ contains a Sylow 2-subgroup of $Soc(G)$, whence K contains a Sylow 2-subgroup of G.

Let $G = G_0 \geqslant G_1 \geqslant \ldots \geqslant G_n = \{e\}$ be a chief series of G. Then $G_i/G_{i+1} = L_{i,1} \times \ldots \times L_{i,k_i}$, where $L_{i,1} \simeq \ldots \simeq L_{i,k_i} \simeq L_i$ and L_i is a simple group. If $i \geqslant 1$, then denote by $\overline{K}_i$ a Carter subgroup of G/G_i (if it exists), and by K_i its complete preimage in G/G_{i+1}. If $i = 0$, the set $\overline{K}_0 = \{e\}$ and $K_0 = G/G_1$ (notice that $\overline{K}_0$ always exists). We say that G satisfies **(E)**, if $\mathrm{Aut}_{K_i}(L_{i,j})$ possesses a Carter subgroup for all i, j.

Theorem 5. (EV 2008)
G possesses a Carter subgroup if and only if G satisfies **(E)**.

Proof. Clearly we need to consider the case, when G possesses a normal subgroup $B = T_1 \times \ldots \times T_n$ such that G/B is nilpotent. Consider the embedding $G \to \mathrm{Aut}_G(T_1) \wr P$ from Theorem 2. Condition **(E)** implies that $\mathrm{Aut}_G(T_1)$ possesses a Carter subgroup K_1. Then $(K_1 \wr P) \cap G$ is a solvable selfnormalizing subgroup of G, hence its Carter subgroup is a Carter subgroup of G. The "only if" part follows immediately from Lemma 1.

Lemma 8.
Let S be a known finite simple group, $S \not\simeq J_1$ and $G = \mathrm{Aut}(S)$. Then G possesses a Carter subgroup.

Now we construct example showing that an extension of a group containing a Carter subgroup, by a group, containing a Carter subgroup, may fail to contain a Carter subgroup.

Consider $L = \mathrm{PSL}_2(3^3) \leftthreetimes \langle \varphi \rangle$, where φ is a field automorphism of $\mathrm{PSL}_2(3^3)$ and $X = L \wr \mathrm{Sym}_2$. Denote by $H = \mathrm{PSL}_2(3^3) \times \mathrm{PSL}_2(3^3)$ a minimal normal subgroup of X and by $M = L_1 \times L_2$. Let $G = (H \leftthreetimes \langle (\varphi, \varphi^{-1}) \rangle) \leftthreetimes \mathrm{Sym}_2$ be a subgroup of X. Then the following statements hold:

1. $G \cap M \trianglelefteqslant G$ contains a Carter subgroup.
2. $G/(G \cap M)$ is nilpotent.
3. G does not contain a Carter subgroup.

Lecture 4. Hall subgroups (Existence).

First we fix notation for this lecture.

Symbol p always denotes a prime, while π is a set of primes. By p' we denote the set of primes not equal to p, and by π' we denote the set of all primes not in π.

$\pi(n)$ is the set of all prime divisors of a positive integer n, for a group G we denote $\pi(|G|)$ by $\pi(G)$. A positive integer n with $\pi(n) \subseteq \pi$ is called a π-number, a group G with $\pi(G) \subseteq \pi$ is called a π-group. Given positive integer n denote by n_π the maximal divisor t of n with $\pi(t) \subseteq \pi$.

One of the first result in the theory of finite groups

Lagrange Theorem
If H is a subgroup of a finite group G, then $|G| = |H| \cdot |G : H|$.

H is a Sylow p-subgroup of G if $\pi(H) \subseteq \{p\}$ and $\pi(\lvert G : H\rvert) \cap \{p\} = \varnothing$	H is a π-Hall subgroup of G if $\pi(H) \subseteq \pi$ and $\pi(\lvert G : H\rvert) \cap \pi = \varnothing$
$\mathrm{Syl}_p(G)$ is the set of Sylow p-subgroups of G	$\mathrm{Hall}_\pi(G)$ is the set of π-Hall subgroups of G
Sylow theorems	Hall properties
E-theorem: $\mathrm{Syl}_p(G) \neq \varnothing$	E_π-property: $\mathrm{Hall}_\pi(G) \neq \varnothing$
C-theorem: members of $\mathrm{Syl}_p(G)$ are conjugate	C_π-property: E_π+members of $\mathrm{Hall}_\pi(G)$ are conjugate
D-theorem: each p-subgroup of G is included in a Sylow p-subgroup	D_π-property: C_π+each π-subgroup of G is included in a π-Hall subgroup

Theorem (P.Hall 1928; P.Hall 1936, S.Chunikhin 1937)
A finite group G is solvable if and only if $G \in D_\pi$ for every set π of primes.

Consider Alt_5. We have $|\mathrm{Alt}_5| = 60$, so, if Alt_5 would possess a $\{3,5\}$-Hall subgroup H, then $|H| = 15$ and so $|\mathrm{Alt}_5 : H| = 4$. Thus Alt_5 would possess a nontrivial permutation representation of degree 4 in spite of the simplicity of Alt_5. So $\mathrm{Alt}_5 \notin E_{\{3,5\}}$.

Consider $\mathrm{GL}_3(2)$. We have $|\mathrm{GL}_3(2)| = 168 = 2^3 \cdot 3 \cdot 7$. Let H_1 be a line-stabilizer and H_2 be a plane-stabilizer. Then $|H_1| = |H_2| = 2^3 \cdot 3 = 24$, so both H_1 and H_2 are $\{2,3\}$-Hall subgroups of $\mathrm{GL}_3(2)$. Clearly H_1 and H_2 are not conjugate, so $\mathrm{GL}_3(2) \in E_{\{2,3\}} \setminus C_{\{2,3\}}$.

Consider Alt_5. Then Alt_4 is a $\{2,3\}$-Hall subgroup of Alt_5 and all subgroups of order 12 are point-stabilizers, so they are conjugate. Alt_5 possesses a subgroup of order 6, while Alt_4 does not, so $\mathrm{Alt}_5 \in C_{\{2,3\}} \setminus D_{\{2,3\}}$.

Lemma 1.
Let A be a normal and H be a π-Hall subgroups of G. Then $H \cap A$ is a π-Hall subgroup of A and HA/A is a π-Hall subgroup of G/A.

Proof. Since $|G : HA|$ divides $|G : H|$ we obtain that $|G : HA|$ is a π'-number and $HA/A \in \mathrm{Hall}_\pi(G/A)$. Since $|HA| = \frac{|H|\cdot|A|}{|H\cap A|}$, it follows that $|HA : H| = |A : (H \cap A)|$. So $|A : (H \cap A)|$ is a π'-number and $H \cap A \in \mathrm{Hall}_\pi(A)$. □

	E_π	C_π	D_π
Normal subgroups	Yes		
Factor groups	Yes		
Extensions			

Schur-Zassenhaus Theorem
Let $A \trianglelefteq G$ and $(|A|, |G : A|) = 1$. Then there exists $H \leqslant A$ such that $AH = G$ and $A \cap H = 1$. Moreover, all such subgroups H are conjugate. Therefore, if G possesses a subnormal series such that all factors are either π- or π'-groups, then $G \in E_\pi$.

Lemma 2.
If $A \trianglelefteq G$ and H/A is a π-subgroup of G/A, then there exists a π-subgroup M of G such that $MA = H$.

Proof. Let M be a minimal subgroup of G with $MA = H$. Then $M \cap A \leqslant \Phi(M)$, in particular, $\pi(M \cap A) \subseteq \pi(M/M \cap A) = \pi(H/A)$ (the inclusion $\pi(\Phi(G)) \subseteq \pi(G)$ follows by Schur-Zassenhaus theorem). □

	E_π	C_π	D_π
Normal subgroups	Yes		
Factor groups	Yes		Yes
Extensions			

If $A \trianglelefteq G$ and H is a subgroup of A such that $H^A = \{H^a \mid a \in A\} = H^G$, then we can apply Frattini argument and obtain that $G = AN_G(H)$.

Extension Lemma
If $A \trianglelefteq G$, $\pi(G/A) \subseteq \pi$, and $M \in \mathrm{Hall}_\pi(A)$, then $H \in \mathrm{Hall}_\pi(G)$ with $H \cap A = M$ exists if and only if $M^A = M^G$. In particular, an extension of a C_π-group by a C_π-group satisfies C_π.

Proof. The "only if" part follows from $G = HA$. If $M^A = M^G$, then we can apply Frattini argument to obtain that $G = N_G(M)A$. In particular, $|G : N_G(A)|$ is a π'-number. Consider the normal series $N_G(M) \trianglerighteq N_A(M) \trianglerighteq M \trianglerighteq 1$. Each section of the series is either a π-group, or a π'-group. So $N_G(M)$ possesses a π-Hall subgroup H by Schur-Zassenhaus Theorem and $M \leqslant H$. Clearly $|G : H|$ is a π'-number, i.e. $H \in \mathrm{Hall}_\pi(G)$ and $H \cap A = M$. □

Corollary
Let G be an almost simple E_π-group with a simple socle S. Then $L \in E_\pi$ for every $S \leqslant L \leqslant G$.

Proof. Since L/S is solvable, it possesses a π-Hall subgroup, so we may assume that L/S is a π-group. Consider $H \in \mathrm{Hall}_\pi(G)$. Hall theorem implies that $L^g \leqslant HS$ for some $g \in G$. Now Extension lemma implies that $H \cap S$ is HS-invariant (and, in particular, L^g-invariant), so $H \cap S$ is included in a π-Hall subgroup of L^g. □

	E_π	C_π	D_π
Normal subgroups	Yes		
Factor groups	Yes		Yes
Extensions		Yes	

Let $\pi = \{2, 3\}$, $G = \mathrm{GL}_3(2) = \mathrm{SL}_3(2)$ be a group of order $168 = 2^3 \cdot 3 \cdot 7$. Then G has exactly two classes of π-Hall subgroups with representatives

$$\begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & * \\ 0 & \boxed{1} \end{pmatrix} \text{ and } \begin{pmatrix} \boxed{1} & * \\ 0 & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}.$$

The map $\iota : x \in G \mapsto (x^t)^{-1}$ is an automorphism of order 2 of G. It interchanges classes of π-Hall subgroups, hence the group $\widehat{G} = G : \langle\iota\rangle$ does not possess a π-Hall subgroup.

	E_π	C_π	D_π
Normal subgroups	Yes		
Factor groups	Yes		Yes
Extensions	No	Yes	

Let $\pi = \{2, 3\}$, $G = \mathrm{GL}_5(2)$, $\iota : x \in G \mapsto (x^t)^{-1}$ and $\widehat{G} = G \leftthreetimes \langle\iota\rangle$. Then $G \in E_\pi$ and, if $H \in \mathrm{Hall}_\pi(G)$, then H is the stabilizer of a series of subspaces $V = V_0 < V_1 < V_2 < V_3 = V$, where V is the natural G-module and $\dim V_k/V_{k-1} \in \{1, 2\}$ for $k = 1, 2, 3$. Therefore G possesses three classes of conjugate π-Hall subgroups:

$$H_1 = \begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & & * \\ & \boxed{1} & \\ 0 & & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}, H_2 = \begin{pmatrix} \boxed{1} & & * \\ & \boxed{\mathrm{GL}_2(2)} & \\ 0 & & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}, H_3 = \begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & & * \\ & \boxed{\mathrm{GL}_2(2)} & \\ 0 & & \boxed{1} \end{pmatrix}.$$

The class containing H_1 is ι-invariant, so Extension Lemma implies that there exists $H \in \mathrm{Hall}_\pi(\widehat{G})$ such that $H \cap G = H_1$. ι permutes classes containing H_2 and H_3. So Extension Lemma implies that both H_2, H_3 are not included in π-Hall subgroups of $\widehat{G}$. Thus $\widehat{G} \in C_\pi$, while its normal subgroup G does not satisfy C_π.

	E_π	C_π	D_π
Normal subgroups	Yes	No	
Factor groups	Yes		Yes
Extensions	No	Yes	

	E_π	C_π	D_π
Normal subgroups	Yes	No	Yes (mod CFSG)
Factor groups	Yes	Yes (mod CFSG)	Yes
Extensions	No	Yes	Yes (mod CFSG)

Theorem 6.
Let G be a group satisfying Theorem 2, $L_1 \times \ldots \times L_k = T \trianglelefteqslant G$, and $P \simeq G/T$ be a π-group. Assume that $\mathrm{Aut}_G(L_1)$ possesses a π-Hall subgroup H_1. Then $H_1 \wr P$ is a π-Hall subgroup of $\mathrm{Aut}_G(L_1) \wr P$ and $G^\varphi \cap (H_1 \wr P)$ is a π-Hall subgroup of G^φ, i.e. $G \in E_\pi$. Moreover each π-Hall subgroup of G^φ can be obtained in this way.

Proof. We identify G with its image under φ to simplify notation. Clearly $|(\mathrm{Aut}_G(L_1) \wr P) : (H_1 \wr P)| = |\,\mathrm{Aut}_G(L_1) : H_1|^k$, so $|(\mathrm{Aut}_G(L_1) \wr P) : (H_1 \wr P)|$ is a π'-number. Thus $H_1 \wr P \in \mathrm{Hall}_\pi(\mathrm{Aut}_G(L_1) \wr P)$. Denote $G \cap (H_1 \wr P)$ by H. Since $T \trianglelefteqslant \mathrm{Aut}_G(L_1) \wr P$, Lemma 1 implies that $H \cap T \in \mathrm{Hall}_\pi(T)$. Since G/T is a π-group, it follows that $\mathrm{Aut}_G(L_1) = L_1 H_1$, so $\mathrm{Aut}_G(L_1) \wr P = T(H_1 \wr P)$ and $G = TH$. Now $|G : H| = |T : (T \cap H)|$, whence $|G : H|$ is a π'-number and so $H \in \mathrm{Hall}_\pi(G)$. Finally, if $H \in \mathrm{Hall}_\pi(G)$, then $H \cap L_1 \in \mathrm{Hall}_\pi(L_1)$ and $\mathrm{Aut}_G(L_1) = L_1 \,\mathrm{Aut}_H(L_1)$, whence $\mathrm{Aut}_H(L_1) \in \mathrm{Hall}_\pi(\mathrm{Aut}_G(L_1))$. $\square$

Corollary. (F.Gross, 1986)
Let $G = G_0 \geqslant G_1 \geqslant \ldots \geqslant G_n = 1$ be an (rc)-series of G. Assume that $\mathrm{Aut}_G(G_i/G_{i+1}) \in E_\pi$ for $i = 0, \ldots n-1$. Then $G \in E_\pi$.

Proof. Let N be a minimal normal subgroup of G such that $G_{n-1} \leqslant N$. Clearly, G/N satisfies the condition of the corollary, so by induction G/N possesses a π-Hall subgroup. So we may assume that G/N is a π-group. It is not difficult to see, that N should be nonsolvable and $C_N(G) = 1$. Now Theorem 6 implies that $G \in E_\pi$. $\square$

"Theorem"
Hall subgroups and conjugacy classes of Hall subgroups in finite simple groups are known.

Lemma 3. (D.Revin, EV, 2011)
Let S be a non-abelian finite simple E_π-group and suppose that $S < G \leqslant \mathrm{Aut}(S)$ and $G \notin E_\pi$. Then $2, 3 \in \pi \cap \pi(S)$ and there exists a 2-element x of G such that $\langle x, S \rangle \notin E_\pi$.

Corollary (D.Revin, EV, 2011)
Let $G = G_0 \geqslant G_1 \geqslant \ldots \geqslant G_n = 1$ be an composition series of a E_π-group G. Then $\mathrm{Aut}_G(G_i/G_{i+1}) \in E_\pi$ for $i = 0, \ldots n-1$.

Proof. Assume by contradiction that there exists i such that $\mathrm{Aut}_G(G_i/G_{i+1}) \notin E_\pi$. Let N be a minimal normal subgroup of G. If $G_i \not\leqslant N$, then we can consider G/N and obtain the claim by induction. By Lemma 3 there exists 2-element $\bar{x} \in \mathrm{Aut}_G(G_i/G_{i+1})$ such that $\langle \bar{x}, G_i/G_{i+1} \rangle \notin E_\pi$. Extension Lemma implies that $\bar{x}$ does not normalizes any π-Hall subgroup of G_i/G_{i+1}. Let x be a preimage of $\bar{x}$ in $N_G(G_i/G_{i+1})$ and assume that x is a 2-element. By Sylow theorem $x \in H$ for some $H \in \mathrm{Hall}_\pi(G)$. By Lemma 1, $\overline{H} = (H \cap G_i)G_{i+1}/G_{i+1}$ is a π-Hall subgroup of G_i/G_{i+1}. Moreover, $x \in N_H(G_i/G_{i+1})$, so $\bar{x}$ normalizes $\overline{H}$, a contradiction. $\square$

Existence criterion for π-Hall subgroups.

$G \in E_\pi$ if and only if for some (hence, for every) (rc)-composition series $G = G_0 \geqslant G_1 \geqslant \ldots \geqslant G_n = 1$, $\mathrm{Aut}_G(G_i/G_{i+1}) \in E_\pi$ for $i = 0, \ldots n-1$.

Corollary.
Let A be a normal subgroup of an E_π-group G. Then for every π-Hall subgroup M/A there exists a π-Hall subgroup H of G such that $M = HA$. In particular, if $G \in C_\pi$, then $G/A \in C_\pi$.

Proof. Let $1 = G_0 < G_1 < \ldots < G_k = G$ be a composition series of G which is a refinement of a chief series of G through A, so $G_n = A$ for some n. Let $1 = M_n/A < M_{n+1}/A < \ldots < M_m/A = M/A$ be an (rc)-series of M/A. Then $1 = G_0 < \ldots < G_n = A = M_n < \ldots < M_m = M$ is an (rc)-series of M. Since M/A is a π-group, $\mathrm{Aut}_M(M_i/M_{i-1}) \in E_\pi$ for each $i > n$. For every nonabelian composition factor G_i/G_{i-1} with $i \leqslant n$ we have that $G_i/G_{i-1} \leqslant \mathrm{Aut}_M(G_i/G_{i-1}) \leqslant \mathrm{Aut}_G(G_i/G_{i-1})$. By Existence criterion, $\mathrm{Aut}_G(G_i/G_{i-1})$ satisfies E_π. By Corollary to Existence criterion we have $\mathrm{Aut}_M(G_i/G_{i-1}) \in E_\pi$. Thus $M \in E_\pi$. Hence there exists a π-Hall subgroup H of M and H is a π-Hall subgroup of G. Since M/A is a π-group, $M = HA$. □

Lecture 5. Hall subgroups (Conjugacy).

Let S be a subnormal subgroup of G. We say that $H \in \mathrm{Hall}_\pi(S)$ is G-induced, if there exists $M \in \mathrm{Hall}_\pi(G)$ such that $M \cap S = H$. Denote by $k_\pi^G(S)$ the number of G-induced classes of conjugate π-Hall subgroups of S. The number $k_\pi^G(G)$ (the number of classes of conjugate π-Hall subgroups of G) we denote $k_\pi(G)$.

Theorem 7. (D.Revin, EV, 2011)
Let π be a set of primes, G a finite almost simple group with nonabelian simple socle S. Then the following statements hold:

(a) if $2 \notin \pi$, then $k_\pi^G(S) \in \{0, 1\}$;

(b) if $3 \notin \pi$, then $k_\pi^G(S) \in \{0, 1, 2\}$;

(c) if $2, 3 \in \pi$, then $k_\pi^G(S) \in \{0, 1, 2, 3, 4, 9\}$.

In particular, $k_\pi^G(S)$ is bounded and, if $G \in E_\pi$, then $k_\pi^G(S)$ is a π-number.

Let S be a simple group possessing $k > 1$ classes of π-Hall subgroups (for example, if $S = \mathrm{GL}_3(2)$ and $\pi = \{2, 3\}$, then $k = 2$). Let p be any prime not lying in $\pi \cap \pi(S)$ (in our example $p > 3$). Then $S \wr \mathbb{Z}_p$ possesses $(k^p - k)/p + k$ classes of π-Hall subgroups. Thus the theorem is not true for arbitrary group.

Corollary. (D.Revin, EV, 2006; D.Revin, EV, 2011)
A normal subgroup of a D_π-group satisfies D_π.

Proof. Let A be a minimal normal non-D_π-subgroup of a D_π-group G and G is of minimal possible order. By Lemma 4.1, $A \in E_\pi$. Each $K \in \mathrm{Hall}_\pi(A)$ is included in some $H \in \mathrm{Hall}_\pi(G)$, so in $H \cap A \in \mathrm{Hall}_\pi(A)$, hence $K = H \cap A$. Thus we need to prove that π-Hall subgroup of A are conjugate.

Notice first that A is a minimal normal subgroup of G. If $N \trianglelefteqslant G$ and $N < A$, then both N and A/N satisfy D_π, whence $A \in C_\pi$.

Thus $A = S_1 \times \ldots \times S_n$, where S_i-s are (nonabelian) simple. Set $k = k_\pi(S_1) = \ldots = k_\pi(S_n)$. Clearly $k_\pi(A) = k^n$, so $k_\pi(A)$ is a π-number.

On the other hand, each $K \in \mathrm{Hall}_\pi(A)$ has the form $K = H \cap A$ for suitable $H \in \mathrm{Hall}_\pi(G)$. So G acts on the classes of conjugate π-Hall subgroups of A transitively (since $G \in D_\pi$) and H is the point stabilizer of $(H \cap A)^A$. In particular, $k_\pi(A)$ divides $|G : H|$ and is a π'-number. So $k_\pi(A) = 1$ and $A \in C_\pi$. □

Theorem 8. (D.Revin, EV, 2010)
If $G \in C_\pi$, $H \in \mathrm{Hall}_\pi(G)$, and $A \trianglelefteqslant G$, then $AH \in C_\pi$.

Proof. Choose A minimal. If $K \in \mathrm{Hall}_\pi(HA)$, then, clearly $HA = KA$. By induction, it follows that A is a minimal normal subgroup of G. Thus $A = S_1 \times \ldots \times S_n$, where S_i-s are (nonabelian) simple. Clearly $N_G(HA) \in C_\pi$, so, by induction, HA is normal in G. In particular, each π-Hall subgroup of G lies in HA. Consider the action of HA on $\{S_1, \ldots, S_n\}$ by conjugation. Let $\Delta_1, \ldots, \Delta_s$ be the orbits of the action and $T_j = \langle \Delta_j \rangle$. If we choose any j and $S_{i,j} \in \Delta_j$, then $k_\pi^{HA}(T_j) = k_\pi^{HA}(S_{i,j})$. Indeed, clearly we may assume that $A = T_j$ and $C_{HA}(A) = 1$ (by taking homomorphic images). Consider the embedding $\varphi : HA \to \mathrm{Aut}_{HA}(S_{i,j}) \wr P$. Clearly a π-Hall subgroup of $S_{i,j}$ is HA-induced if and only if it is $\mathrm{Aut}_{HA}(S_{i,j})$-induced. Theorem 6 implies that the same statement holds for π-Hall subgroups of A.Thus $k_\pi^G(A) = k_\pi^{HA}(A) = k_\pi^{HA}(T_j)^s = k_\pi^{HA}(S)^s$, and Theorem 7 implies that $k_\pi^G(A)$ is a π-number. On the other hand, G acts transitively on the set of classes of conjugate π-Hall subgroups in A, while HA stabilizes such classes. Therefore $k_\pi^G(A)$ is a π'-number, so $k_\pi^G(A) = 1$. □

Corollary. (Conjugacy criterion)
Let π be a set of primes and let A be a normal subgroup of G. Then $G \in C_\pi$ if and only if $G/A \in C_\pi$ and, for a π-Hall subgroup K/A of G/A its complete preimage K satisfies C_π.

Proof. Assume that $G \in C_\pi$. Then Corollary to Existence criterion implies that $G/A \in C_\pi$ and that there exists $H \in \mathrm{Hall}_\pi(G)$ such that $K = HA$. Now Theorem 8 implies $HA \in C_\pi$.

Assume that $G/A \in C_\pi$ and for some $K/A \in \mathrm{Hall}_\pi(G/A)$ we have $K \in C_\pi$. If $H \in \mathrm{Hall}_\pi(K)$, then clearly $H \in \mathrm{Hall}_\pi(G)$, so $G \in E_\pi$. Assume that $H, M \in \mathrm{Hall}_\pi(G)$. We may assume that both $H, M \in \mathrm{Hall}_\pi(K)$, since $G/A \in C_\pi$. Now the claim follows from $K \in C_\pi$. □

Notice that in the proof of Theorem 8 we obtain the following statement.

Lemma 1.
If G satisfies the condition of Theorem 2 and G/T is a π-group, then $G \in C_\pi$ if and only if $\mathrm{Aut}_G(T_1) \in C_\pi$.

Let

$$1 = G_n < G_{n-1} < \ldots < G_0 = G$$

be a composition series of a finite group G which is a refinement of a chief series

$$1 = G_n < G_{k_m} < \ldots < G_0 = G.$$

Set $\overline{H}_0 = 1$ and $H_0 = G_0 = G$. Assume that we have constructed $\overline{H}_j \in \mathrm{Hall}_\pi(G/G_{k_j})$ and H_j is its complete preimage in G. We check, if $\mathrm{Aut}_{H_j}(G_{k_{j+1}-1}/G_{k_{j+1}}) \in C_\pi$.

If not, then $G \notin C_\pi$.

If yes, then $G/G_{k_{j+1}} \in C_\pi$ and we take $\overline{H}_{j+1}$ to be any π-Hall subgroup of $G/G_{k_{j+1}}$.

Now H_n is a π-Hall subgroup of G and, if H_n exists, then $G \in C_\pi$.

Recall that a subgroup H of a group G is called pronormal (we write H prn G), if for every $g \in G$ subgroups H and H^g are conjugate in $\langle H, H^g\rangle$.

Theorem 9. (D.Revin, EV, 2012)
Hall subgroups of finite simple groups are pronormal.

Let S be a finite group possessing nonconjugate Hall subgroups H, K (for example, we may choose $S = \mathrm{GL}_3(2)$ and H, K to be plane and line stabilizers of the natural module). Choose any prime p not dividing $|H| = |K|$ (in example, any $p > 3$). Now consider

$$G = S \wr \mathbb{Z}_p = (S_1 \times \ldots \times S_p) \rtimes \mathbb{Z}_p.$$

Clearly, $H_1 \times K_2 \times \ldots \times K_p$ and $K_1 \times \ldots \times K_{p-1} \times H_p$ are conjugate in G, while they are not conjugate in their common normal closure $S_1 \times \ldots \times S_p$. Thus Hall subgroups are not pronormal in general.

Corollary 1. (D.Revin, EV, 2013)
Let H be a π-Hall subgroup of a C_π-group G. Assume that $M \leqslant G$ is chosen so that $H \leqslant M$. Then $M \in C_\pi$.

Proof. By induction we may assume that $G = HA$, where A is a minimal normal subgroup of G. Now the claim is immediate from Theorem 9. □

Corollary 2. (D.Revin, EV, 2014) Frattini argument for Hall subgroups
Let A be a normal subgroup of a finite E_π-group G. Then there exists $H \in \mathrm{Hall}_\pi(A)$ such that $G = AN_G(H)$.

Proof. By induction we may assume that A is a minimal normal subgroup of G. Also by induction we may assume that G satisfies Theorem 2 with $A = T = L_1 \times \ldots \times L_k$. By Corollary to Lemma 4.3 we obtain that $\mathrm{Aut}_G(L_1) \in E_\pi$. If $M \in \mathrm{Hall}_\pi(\mathrm{Aut}_G(L_1))$ and $H_1 = M \cap L_1$, then for $H = H_1 \times \ldots \times H_k \leqslant A$ we have $\mathrm{Aut}_G(L_1) \wr P = AN_{\mathrm{Aut}_G(L_1) \wr P}(H)$. Whence $G = AN_G(H)$. □

Corollary 3. (D.Revin, EV, to appear)
Let A be a normal subgroup of an E_π-group G. Then A possesses $H \in \mathrm{Hall}_\pi(G)$ such that H prn G. In particular, an E_π-group possesses a pronormal π-Hall subgroup.

Proof. Again it is not difficult to see, that in minimal counterexample A should be minimal normal subgroup. Now the claim follows from Theorem 9 and Corollary 2. □

Notice that Corollary 1 can be reformulated in equivalent form

Corollary 1'.
Let H be a π-Hall subgroup of a C_π-group G. Then H prn G.

Proof. Assume that every $M \leqslant G$ with $H \leqslant M$ satisfies C_π. Then for every $g \in G$, $\langle H, H^g\rangle \in C_\pi$, so H, H^g are conjugate in $\langle H, H^g\rangle$. Conversely, if H prn G and $H \leqslant M$, then for every $K \in \mathrm{Hall}_\pi(G)$ there exists $g \in G$ such that $H^g = K$. Now H, H^g are conjugate in $\langle H, H^g\rangle \leqslant M$. □

Theorem 10. (N.Ch.Manzaeva, to appear)
Let H be a π-Hall subgroup of a D_π-group G. Then for every $M \leqslant G$ such that $H \leqslant M$ we have $M \in D_\pi$

Notice that Theorem 10 can be formulated in an equivalent form (like Corollaries 1 and 1') if we introduce the following definition. A subgroup H is strongly pronormal in G if, for every $K \leqslant H$ and $g \in G$ there exists $x \in \langle K^g, H\rangle$ such that $K^{gx} \leqslant H$. So the equivalent form of Theorem 10 is:

Theorem 10'.
π-Hall subgroups in finite D_π-groups are strongly pronormal.

It is not difficult to construct example of pronormal but not strongly pronormal subgroups in Sym_n. Carter subgroups are pronormal, but they can be non strongly pronormal even in solvable groups.

Problem 1.
Let $G = G_0 \geqslant G_1 \geqslant \ldots \geqslant G_n = 1$ be an composition series of a E_π-group G and $H \in \mathrm{Hall}_\pi(G)$. Does $\mathrm{Aut}_H(G_i/G_{i+1})$ lies in $\mathrm{Hall}_\pi(\mathrm{Aut}_G(G_i/G_{i+1}))$ for $i = 0, \ldots n-1$?

Problem 2. (Question 17.45(b), "Kourovka notebook")
In a finite simple group, are Hall subgroups always strongly pronormal?

We say that a finite group G possesses a <u>Sylow tower of complexion</u> $(p_1, \ldots, p_n)$, if $\pi(G) = \{p_1, \ldots, p_n\}$ and there exists a normal series

$$G = G_0 > G_1 > \ldots > G_{n-1} > G_n = 1$$

such that G_{i-1}/G_i is isomorphic to a Sylow p_i-subgroup of G. It is not difficult to see, that π-Hall subgroups possessing Sylow towers of the same complexion are conjugate. So if a Hall subgroup possesses a Sylow tower, then it is pronormal.

Problem 3. (Question 17.45(c), "Kourovka notebook")
In a finite group, is a Hall subgroup with a Sylow tower always strongly pronormal?

Problem 4. (Question 18.32, "Kourovka notebook")
Is every Hall subgroup of a finite group pronormal in its normal closure.

Problem 5.
In a finite group is a pronormal Hall subgroup always strongly pronormal?

Problem 6.
Let A be a normal and H be a π-Hall subgroups of G. Assume also that H prn G. Is $H \cap A$ pronormal in A? $(H \cap A)^G = (H \cap A)^A$?

Problem 7. (Question 18.38, "Kourovka notebook")
Does the inclusion $E_{\pi_1} \cap E_{\pi_2} \subseteq E_{\pi_1 \cap \pi_2}$ hold for arbitrary sets of primes π_1 and π_2? (A.V.Zavarnitsine)

Problem 8.
Does G always satisfy E_π, if G satisfies $E_{\{r,s\}}$ for every $r, s \in \pi$?

Lecture 6. Base size (Preliminaries)

Assume that G acts transitively on X, H is the stabilizer of a point x (usually $X = \{1, \ldots, n\}$ and $x = 1$), and $H_G = \cap_{g \in G} H^g$ is the kernel of the action. $\{x_1, \ldots x_k\} \subset X$ is called a <u>base</u> (we denote it by $\mathrm{Base}(G)$) if $G_{x_1} \cap \ldots \cap G_{x_k} = H_G$ (clearly X is always a base). Minimal k is called a <u>base size</u> of G and is denoted $b(G)$.

Example 1.
1. Let $G = \mathrm{GL}(V)$ and $X = V$. Then subset $\{v_i \mid i \in I\}$ is a base if and only if it spans V. Moreover $b(G) = \dim V$ and a minimal (by inclusion) base is a basis of V.

2. Consider $A = F_q$ and $B = F_q^*$. Then B acts on A by multiplication. Moreover, $A \rtimes B$ acts on $X = A = F_q$ by $ab : \alpha \mapsto b \cdot (\alpha + a)$. The action is transitive and B is the point stabilizer. Clearly a subset Y of X is a base if and only if $|Y| \geqslant 2$.

Base size problem
Given G acting transitively on X find $b(G)$.

Notice that Lagrange theorem in terms of action can be formulated as $|G| = |G_x| \cdot |x^G|$.The orbit x^G is called <u>regular</u>, if $|x^G| = |G|$, and the point x is called a <u>G-regular</u> point. By Lagrange theorem x is a G-regular point if and only if $G_x = 1$ (in particular, $\{x\}$ is a base).

We always assume that G acts on X^m by

$$g : (x_1, \ldots, x_m) \mapsto (x_1^g, \ldots, x_k^g).$$

Clearly $\{x_1, \ldots, x_k\}$ is a base if and only if $(x_1, \ldots, x_k)$ is a G/H_G-regular point (in X^k).

Base size problem
Given G acting transitively on X with kernel H_G find the minimal k such that X^k possesses a G/H_G-regular point.

Denote by $\mathrm{Reg}(G, m)$ the number of G/H_G-regular orbits in X^m. Clearly $\mathrm{Reg}(G, m) > 0$ if and only if $b(G) \leqslant m$.

Assume that H is the stabilizer of $x_1 \in X$. Since G acts transitively, for every $x_2 \in X$ there exists $g \in G$ such that $x_1^g = x_2$. Easy calculations show that H^g is the stabilizer of x_2. Now consider $\{x_1, \ldots, x_k\} \subseteq X$ and choose $g_i \in G$ so that $x_1^{g_i} = x_i$. By definition, $\{x_1, \ldots, x_k\}$ is a base if and only if

$$H^{g_1} \cap \ldots \cap H^{g_k} = H_G.$$

Base size problem
Given G and $H \leqslant G$ find the minimal k such that there exist $g_1, \ldots, g_k$ with $H^{g_1} \cap \ldots \cap H^{g_k} = H_G$.

A base of G acting by right multiplication on G/H we denote by $Base_G(H)$, the base size by $b_H(G)$, and the number of G/H_G-regular orbits on $(G/H)^m$ by $\mathrm{Reg}_H(G, m)$.

Clearly, if $k = b_H(G)$, then

$$|G/H_G| \leqslant |G : H| \cdot (|G : H| - 1) \cdot \ldots \cdot (|G : H| - k + 1) < |G : H|^k.$$

Consider $K \leqslant H \leqslant G$. If $K \geqslant H_G$ (this condition holds, for example, when $H_G = 1$, in particular, when G is almost simple and H does not contain the minimal normal subgroup of G), then $b_K(G) \leqslant b_H(G)$. Indeed, if $H^{x_1} \cap \ldots \cap H^{x_m} = H_G$, then $K^{x_1} \cap \ldots \cap K^{x_m} = H_G = K_G$. However, if $K \not\geqslant H_G$, then the inequality $b_K(G) \leqslant b_H(G)$ may fail. For example consider $G = A_4$, $H = \langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle$, and $K = \langle (1, 2)(3, 4) \rangle$. $H \trianglelefteq G$ and $|G/H| = 3$, so $b_H(G) = 1$. On the other hand, $K_G = 1$, so the embedding $G \to \mathrm{Sym}(G/K)$ is injective, in particular, the order of the image of G equals 12. Now $|G : K| = 6$, so G cannot have a regular orbit on G/K, thus $b_K(G) \geqslant 2$.

We investigate Base size problem for solvable point stabilizer.

Problem
Let H be a maximal solvable subgroup of G. Denote by $S(G)$ the solvable radical of G. Does there exists an absolute constant c such that for some $x_1, \ldots, x_c \in G$ the equality $H^{x_1} \cap \ldots \cap H^{x_c} = S(G)$? In other words, does there exists an absolute constant c such that $b_H(G) \leqslant c$ for any G and every maximal solvable subgroup H of G?

Lemma 1.
Assume that G satisfies condition of Theorem 2. Assume also that G/T is solvable. Then

(a) there exists a maximal solvable subgroup S of G such that $G = ST$;

(b) if we choose a maximal solvable subgroup S of G such that $G = ST$, then $\overline{G} = \mathrm{Aut}_G(L_1) \wr P$ possesses a maximal solvable subgroup $\overline{S}$ such that $S \leqslant \overline{S}$ and $\overline{G} = \overline{S}T$.

Proof. Let K be a minimal subgroup of G with $KT = G$. As we noted earlier, $K \cap T \leqslant \Phi(K)$, so K is solvable. If we choose $S \leqslant G$ such that S is maximal subject to be solvable and to $G = ST$, then S is a maximal solvable subgroup of G with $G = ST$. . Consider $\overline{S} = \mathrm{Aut}_S(L_1) \wr P$. Clearly $\overline{S}$ is solvable, $\overline{S} \cap G^\varphi = S^\varphi$, and $T\overline{S} = \overline{G}$. It is not difficult to see that $N_{\overline{G}}(\overline{S}) = \overline{S}$, so $\overline{S}$ is a maximal solvable subgroup of $\overline{G}$. □

Let G be a subgroup of $\mathrm{Sym}(X)$. A partition $X_1 \sqcup \ldots \sqcup X_k$ is called asymmetric for G (or G-asymmetric), if only the identity element of G stabilizes the partition, i.e. if $(X_i)g = X_i$ for each $i = 1, \ldots, k$, then $g = 1$.The minimal k such that there exists G-asymmetric partition with k subsets is called the G-asymmetric partition degree. We always consider a partition $X_1 \sqcup \ldots \sqcup X_k = X$ as a colouring to k colours.

Consider $G = \mathrm{Sym}_4 \wr \mathrm{Sym}_2$ as a (nonprimitive) subgroup of Sym_8 and assume that its normal subgroup $\mathrm{Sym}_4 \times \mathrm{Sym}_4$ stabilizes $\{1, 2, 3, 4\} \cap \{5, 6, 7, 8\}$. Then a G-asymmetric partition is equal to $\{1, 5\} \sqcup \{2, 6\} \sqcup \{3, 7\} \sqcup \{4\} \sqcup \{8\}$.

Lemma 2. (A.Seress, 1996)
Let G be a solvable permutation group. Then there exists a G-asymmetric partition of degree at most 5.

Proof. In 1983 D.Gluck proved, that for every primitive permutation group G, with a few exceptions, there exists a subset Δ of X such that set-stabilizer of Δ in G is trivial. Exceptions: Frobenius groups of degree 3, 5, or 7; $\mathrm{Alt}_4, \mathrm{Sym}_4 \leqslant \mathrm{Sym}_4$; $\mathrm{A\Gamma L}_1(2^3) \leqslant \mathrm{Sym}_8$; $\mathrm{AGL}_2(3^2) \leqslant \mathrm{Sym}_9$.

So if G is a primitive permutation group, then the degree of G-asymmetric partition is at most 4. Clearly we may assume that G is transitive of degree n and $\{1,\ldots,n\}=X$. Define a structure tree for G as a rooted tree with levels $T_0,T_1,\ldots,T_m$ such that the root $T_0=\{X\}$, the leaves (nodes on level T_m) are elements of X, and internal nodes of T correspond to certain blocks of imprimitivity in X. The blocks on a fixed level T_i define a partition of X into a block system. We also require that for each non-leaf node x the children of x are a partition of x and the stabilizer of x acts primitively on the children of x. Note that the action of G can be extended naturally to the structure tree, and G acts transitively on each level.

If we consider $G=\mathrm{Sym}_4\wr\mathrm{Sym}_2$, then $T_0=\{1,2,3,4,5,6,7,8\}$; T_1 consists of $\{1,2,3,4\}$ and $\{5,6,7,8\}$, and T_2 is the last level, consists of 1, 2, 3, 4, 5, 6, 7, and 8.

We define a colouring $F:T\to F_5$. Let the colour of the root be 0. Recursively suppose that the colouring of $T_0,T_1,\ldots,T_i$ is already defined and let $x\in T_i$. Since G_x acts primitively on the set $[x]$ of children of x, there exists colouring into $k(x)\leqslant 4$ colours for the children of x. The we colour $[x]$ with colours $F(x)+1,\ldots,F(x)+k(x)$.

It is clear that if $g\in G$ preserves colouring of leaves, then $g=e$. □

Corollary 1.
If $|G|$ is odd, then the G-asymmetric partition degree is not greater than 2.

Proof. By Feit-Thompson Odd Order Theorem G is solvable. By Gluck's theorem, if G is a primitive solvable group of odd order, then there exists a G-asymmetric partition $Y\sqcup Z=X$. Moreover, $|X|$ is odd, hence, without loss of generality we may assume $|Y|<|Z|$. Now we can define a colouring $F:T\to F_2$ recursively on i-th level by the following rule: $[x]=Y(x)\sqcup Z(x)$, where $|Y(x)|<|Z(x)|$, and vertices from $Y(x)$ have colour $F(x)$, while vertices from $Z(x)$ have colour $F(x)+1$. Clearly we can obtain the colouring of T in a unique way from colouring of vertices, thus only the identity preserves the partition. □

Corollary 2. (R.Kurmazov, 2013)
If G is nilpotent, then G-asymmetric partition of degree is at most 3.

Proof. Nilpotent group is primitive if and only if it is cyclic of prime degree, so a primitive nilpotent group has an asymmetric partition of degree 2. Now we repeat the proof of Lemma 2. □

Lemma 3.
Let G be a group and let M be a solvable subgroup of Sym_n. Assume that there exists k such that for every maximal solvable subgroup T of G the inequality $\mathrm{Reg}_T(G,k)\geqslant 5$ holds. Then for every maximal solvable subgroup S of $G\wr M$ we have $\mathrm{Reg}_S(G\wr M,k)\geqslant 5$.

Proof. We assume that $G\wr M$ is a counter example of minimal order. Clearly $S(G)=1$. Consider a solvable subgroup S of $G\wr M=(G_1\times\ldots\times G_n)\rtimes M$. By induction we may assume that $(G_1\times\ldots\times G_n)\cdot S=G\wr M$ and M acts transitively on $\{1,\ldots,n\}$. Denote by S_i the natural projection of $S\cap(G_1\times\ldots\times G_n)$ on G_i. In view of the maximality of S we have $S\cap(G_1\times\ldots\times G_n)=S_1\times\ldots\times S_n$ and each S_i is a maximal solvable subgroup of G_i. Clearly $N_{G_1\times\ldots\times G_n}(S_1\times\ldots\times S_n)=N_{G_1}(S_1)\times\ldots\times N_{G_n}(S_n)=S_1\times\ldots\times S_n$.

Denote by Ω_i the set $\{S_i^x\mid x\in G_i\}$ so that G_i acts on Ω_i by conjugation. Since $N_{G_i}(S_i)=S_i$, it follows that S_i is the point stabilizer under this action and the action is equivalent to the action of G_i on G_i/S_i by right multiplication. Set $\Omega=\Omega_1\times\ldots\times\Omega_n$.

For every $x\in G\wr M$ and for every i we have $S_i^x\leqslant G_j$ for some j. We show that

$$\text{if } S_i^x\leqslant G_j \text{ then } S_i^x\in S_j^{G_j}\text{, i.e., there exists } y\in G_j \text{ such that } S_j^y=S_i^x. \tag{1}$$

Since $(G_1\times\ldots\times G_n)\rtimes M=(G_1\times\ldots\times G_n)S$, it follows that there exists $s\in S$ with $G_i^s=G_j$. We also have $S_i^s=S_j$, since $S\cap(G_1\times\ldots\times G_n)$ is normal in S. Thus $S_i^x=S_j^{s^{-1}x}$. Now $s^{-1}x=g_1\cdot\ldots\cdot g_n\cdot h$, where $g_i\in G_i$ for $i=1,\ldots,n$ and $h\in M$. Since M permutes the G_i-s, it follows that for every $i=1,\ldots,n$, either $G_i^h\cap G_i=\{e\}$, or h centralizes G_i. Thus we obtain that $S_j^{s^{-1}x}=S_j^{g_j}$. So $G\wr M$ acts by conjugation on Ω and S is the stabilizer of the point $(S_1,\ldots,S_n)$. Therefore we need to show that Ω^k possesses at least 5 $(G\wr M)$-regular orbits.

The conditions of the lemma imply that there exist G_1-regular points $\omega_1,\ldots,\omega_5\in\Omega_1^k$ lying in distinct G_1-orbits. If we choose $h_1=e,h_2,\ldots,h_n\in M$ so that $G_1^{h_i}=G_i$, then $\omega_1^{h_i},\ldots,\omega_s^{h_i}\in\Omega_i^k$ are G_i-regular points, and (1) implies that they are in distinct G_i-orbits. We set $\omega_{i,j}=\omega_i^{h_j}$.

By Lemma 2 there exists an asymmetric partition $P_1\sqcup P_2\sqcup P_3\sqcup P_4\sqcup P_5=\{1,\ldots,n\}$ for M. We choose $\omega=(\omega_{i_1,1},\ldots,\omega_{i_n,n})$ so that $i_t=i_j$ if and only if t,j lie in the same P_m. Now we show that $\omega\in\Omega^k$ is a $(G\wr M)$-regular point. Indeed, consider $g=(g_1\ldots g_n)h$, where $g_i\in G_i$ for $i=1,\ldots,n$ and $h\in M$, and assume that $\omega^g=\omega$. It follows that $\omega^{h^{-1}}=\omega^{(g_1\cdots g_n)}$, i.e.,

$$(\omega_{i_1,1},\ldots,\omega_{i_n,n})^{h^{-1}}=(\omega_{i_{(1^h)},1},\ldots,\omega_{i_{(n^h)},n})=(\omega_{i_1,1}^{g_1},\ldots,\omega_{i_n,n}^{g_n}).$$

Therefore $\omega_{i_{(j^h)},j}$ and $\omega_{i_j,j}$ are in the same G_j-orbit, i.e., $i_{(j^h)} = i_j$. By construction, j^h and j are in the same P_m. Whence h stabilizes the partition $P_1 \sqcup P_2 \sqcup P_3 \sqcup P_4 \sqcup P_5$ and $h = e$. We obtain that $(\omega_{i_1,1}, \ldots, \omega_{i_n,n}) = (\omega_{i_1,1}^{g_1}, \ldots, \omega_{i_n,n}^{g_n})$. By construction, $\omega_{i_j,j}$ is a G_j-regular point for every $j = 1, \ldots, n$, so $g_1 = \ldots = g_n = e$, i.e., $g = e$ and $\omega \in \Omega^k$ is a $(G \wr M)$-regular point.

We remain to prove that there exists at least 5 $G \wr M$-regular points in Ω^k.

Consider two $(G \wr M)$-regular points $\omega = (\omega_{i_1,1}, \ldots, \omega_{i_n,n})$ and $\tau = (\omega_{j_1,1}, \ldots, \omega_{j_n,n})$. If $i_1 \neq j_1$, then it follows that $\omega_{i_1,1}$ and $\omega_{j_1,1}$ are in distinct G_1-orbits. Without loss of generality we may assume that $i_1 \in P_1$ and $j_1 \in P_2$. Then ω and τ can lie in the same $G \wr M$ orbit only if there exist $h \in M$ such that $P_1^h = P_2$. Consider the induced action of M on $\{P_1, P_2, P_3, P_4, P_5\}$ (this action can be trivial). It the action is intransitive, then the image of M lies either in $\mathrm{Sym}_3 \times \mathrm{Sym}_2$, or in $\mathrm{Sym}_4 \times \mathrm{Sym}_1$. Now we choose $\omega^{(t)} = (\omega_{i_1^{(t)},1}, \ldots, \omega_{i_n^{(t)},n})$ so that $i_j^{(t)} = \lambda_k^{(t)}$, if $j \in P_k$, where $\lambda^{(1)} = (1,2,3,4,5)$, $\lambda^{(2)} = (1,2,4,3,5)$, $\lambda^{(3)} = (1,2,5,3,4)$, $\lambda^{(4)} = (1,3,4,2,5)$, $\lambda^{(5)} = (1,3,5,2,4)$ in the first case and $\lambda^{(1)} = (1,2,3,4,5)$, $\lambda^{(2)} = (1,2,3,5,4)$, $\lambda^{(3)} = (1,2,5,4,3)$, $\lambda^{(4)} = (1,5,3,4,2)$, $\lambda^{(5)} = (5,2,3,4,1)$ in the second case. If the image of M is transitive, then it lies in $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$. There exist $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}, \lambda(4), \lambda^{(5)} \in \{1,2,3,4,5\}^5$ that lie in distinct $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$-regular orbits. Now we construct $\omega^{(1)}, \ldots, \omega^{(5)}$ in the same way. By construction in all cases $\omega^{(1)}, \ldots, \omega^{(5)}$ lie in distinct $G \wr M$-regular orbits. ☐

Lemma 4.
Let G be a transitive subgroup of Sym_n. Denote $X = \{1, \ldots, n\}$. Let H be the stabilizer of 1 in G.

(a) $(1, i_2, \ldots, i_k)$ and $(1, j_2, \ldots, j_k)$ are in the same G-orbit if and only if $(i_2, \ldots, i_k)$ and $(j_2, \ldots, j_k)$ are in the same H-orbit;

(b) every G-orbit of X^k contains an element $(1, i_2, \ldots, i_k)$;

(c) $(1, i_2, \ldots, i_k)$ is a G-regular point if and only if $(i_2, \ldots, i_k)$ is an H-regular point;

(d) the number of G-orbits in X^k is equal to the number of H-orbits in $(X \setminus \{1\})^{k-1}$;

Notice that we cannot substitute the condition $\mathrm{Reg}_T(G,k) \geqslant 5$ by $\mathrm{Reg}_T(G,k) \geqslant 1$.

Example 2. (V.I.Zenkov, 2009)
Consider $G = \mathrm{Sym}_5$, $M = \mathrm{Sym}_2$. Then for every solvable $T \leqslant G$ we have $\mathrm{Reg}_T(G,4) \geqslant 1$. However $\mathrm{Reg}_{\mathrm{Sym}_4 \wr \mathrm{Sym}_2}(G \wr M, 4) = 0$.

Lecture 7. Base size (Reduction).

Recall that we investigate Base size problem for solvable point stabilizer.

Problem
Let H be a maximal solvable subgroup of G. Denote by $S(G)$ the solvable radical of G. Does there exists an absolute constant c such that for some $x_1, \ldots, x_c \in G$ the equality $H^{x_1} \cap \ldots \cap H^{x_c} = S(G)$? In other words, does there exists an absolute constant c such that $b_H(G) \leqslant c$ for any G and every maximal solvable subgroup H of G?

Theorem 11.
Let G be a group and let

$$\{e\} = G_n < G_{n-1} < G_{n-2} < \ldots < G_0 = G$$

be an (rc)-series of G. Assume that there exists k such that for every nonabelian G_{i-1}/G_i and for every solvable subgroup T of $\mathrm{Aut}_G(G_{i-1}/G_i)$ the inequality $\mathrm{Reg}_T(\mathrm{Aut}_G(G_{i-1}/G_i), k) \geqslant 5$ holds.

Then, for every maximal solvable subgroup S of G, we have $b_S(G) \leqslant k$.

Proof. Assume that the claim is false and G is a counter example of minimal order. Fix a maximal solvable subgroup S of G with $b_S(G) > k$.

Clearly $S(G) = 1$. Consider the generalized Fitting subgroup $F^*(G)$ of G. Since $S(G) = 1$, we obtain that $F^*(G) = L_1 \times \ldots \times L_n$ is a product of nonabelian simple groups and $C_G(F^*(G)) = 1$. If $F^*(G)S \neq G$, then, in view of the minimality of G, there exist $x_1, \ldots, x_k \in F^*(G)S$ such that $S^{x_1} \cap \ldots \cap S^{x_k} = S(F^*(G)S) = 1$. So $G = F^*(G)S$. Moreover, since $L_1, \ldots, L_n$ are nonabelian simple, G acting by conjugation permutes the elements of $\{L_1, \ldots, L_n\}$.

Set $E_1 := \langle L_1^S \rangle$ and $E_2 = \langle L_i \mid L_i \notin \{L_1^s \mid s \in S\} \rangle$. Then $F^*(G) = E_1 \times E_2$ and E_1 and E_2 are S-invariant subgroups. There exists a homomorphism $G \to G/C_G(E_1) \times G/C_G(E_2)$, such that the image of G is a subdirect product of $G/C_G(E_1)$ and $G/C_G(E_2)$, while the kernel is equal to $C_G(E_1) \cap C_G(E_2) = C_G(F^*(G)) = 1$. Denote the projections of G onto $G/C_G(E_1)$ and $G/C_G(E_2)$ by π_1 and π_2 respectively. Then $G^{\pi_1} = E_1(S^{\pi_1})$ and $G^{\pi_2} = E_2(S^{\pi_2})$ (we identify $E_i^{\pi_i}$ with E_i since $E_i^{\pi_i} \simeq E_i$).

Suppose that $E_1 \neq F^*(G)$. Then, by induction for each $i \in \{1,2\}$ there exist elements $x_{1,i},\ldots,x_{k,i}$ of $E_i(S^{\pi_i})$ such that

$$(S^{\pi_i})^{x_{1,i}} \cap \ldots \cap (S^{\pi_i})^{x_{k,i}} = \{e\}. \tag{2}$$

Since $G^{\pi_i} = E_i(S^{\pi_i})$, we may assume that $x_{1,i},\ldots,x_{k,i}$ are in E_i. Consider $x_1 = x_{1,1}x_{1,2},\ldots,x_k = x_{k,1}x_{k,2}$. In view of (2) and $\mathrm{Ker}(\pi_1) \cap \mathrm{Ker}(\pi_2) = \{e\}$, we have $S^{x_1} \cap \ldots \cap S^{x_k} = \{e\}$.

Therefore $E_1 = F^*(G)$ and S acts transitively on $\{L_1,\ldots,L_n\}$. By Lemma 6.1 we may assume that $G = \mathrm{Aut}_G(L_1) \wr S$. Now the claim follows from Lemma 6.3. □

Example 1. (V.I.Zenkov, EV, 2009)
Consider $G = \mathrm{Sym}_8$ and $S = \mathrm{Sym}_4 \wr \mathrm{Sym}_2$. Then $\mathrm{Reg}_S(G,4) = 0$ and $\mathrm{Reg}_S(G,5) = 600$.

Lemma 1.
Let G be a transitive permutation group acting on $\Omega = \{1,\ldots,n\}$ and let the stabilizer S of 1 be solvable. Assume that $k = \max\{b(G), 6\}$. Then $\mathrm{Reg}(G,k) \geqslant 5$.

Proof. By Lemma 6.4, S acts on $\Theta = \Omega \setminus \{1\}$ and the number of G-regular orbits on Ω^k is equal to the number of S-regular orbits on Θ^{k-1}. Thus we need to prove that $\mathrm{Reg}(S,k-1) \geqslant 5$, where S acts on Θ. Since $k \geqslant b(G)$, Lemma 6.4(c) implies that there exist $\theta_1,\ldots,\theta_{k-1} \in \Theta$ such that $(\theta_1,\ldots,\theta_{k-1})$ is an S-regular point in Θ^{k-1}.

Consider $\Delta = \{\theta_1,\ldots,\theta_{k-1}\}$, let T be the setwise stabilizer of Δ in S, i.e., $T = \{x \in S \mid \Delta^x = \Delta\}$. It is clear that $(\theta_{1^\sigma},\ldots,\theta_{(k-1)^\sigma})$ is an S-regular point for every $\sigma \in \mathrm{Sym}_{k-1}$. Moreover if $\sigma,\tau \in \mathrm{Sym}_{k-1}$, then $(\theta_{1^\sigma},\ldots,\theta_{(k-1)^\sigma})$ and $(\theta_{1^\tau},\ldots,\theta_{(k-1)^\tau})$ are in the same S-orbit if and only if there exists $x \in T$ such that $(\theta_{1^\sigma},\ldots,\theta_{(k-1)^\sigma})^x = (\theta_{1^\tau},\ldots,\theta_{(k-1)^\tau})$. Consider the restriction homomorphism $\varphi : T \to \mathrm{Sym}(\Delta)$. Since $(\theta_1,\ldots,\theta_{k-1})$ is an S-regular point (and so a T-regular point), it follows that $Ker(\varphi) = \{e\}$, i.e., φ is injective.

Assume that $k \geqslant 9$ first. Consider an asymmetric partition $P_1 \sqcup P_2 \sqcup P_3 \sqcup P_4 \sqcup P_5 = \{\theta_1,\theta_2,\ldots,\theta_{k-1}\}$ for T^φ. Without loss of generality we may assume that $|P_1| \geqslant |P_2| \geqslant |P_3| \geqslant |P_4| \geqslant |P_5|$. Since $k \geqslant 9$, it follows that either $|P_1| \geqslant 3$, or $|P_1| = |P_2| = |P_3| = 2$.

If $|P_1| \geqslant 3$, then we may assume that $\theta_1,\theta_2,\theta_3 \in P_1$. In this case for every distinct $\sigma,\tau \in \mathrm{Sym}_3$, $(\theta_{1^\sigma},\theta_{2^\sigma},\theta_{3^\sigma},\theta_4\ldots,\theta_{k-1})$ and $(\theta_{1^\tau},\theta_{2^\tau},\theta_{3^\tau},\theta_4,\ldots,\theta_{k-1})$ are in distinct T^φ-orbits, thus these points are in distinct T-orbits, and so in distinct S-orbits. So, in this case $\mathrm{Reg}(S,k-1) \geqslant |\mathrm{Sym}_3| = 6$. If $|P_1| = |P_2| = |P_3| = 2$, then we may assume that $\theta_1,\theta_2 \in P_1$, $\theta_3,\theta_4 \in P_2$, and $\theta_5,\theta_6 \in P_3$. In this case for every distinct $\sigma,\tau \in \mathrm{Sym}(\{1,2\}) \times \mathrm{Sym}(\{3,4\}) \times \mathrm{Sym}(\{5,6\})$, $(\theta_{1\sigma},\theta_{2\sigma},\theta_{3\sigma},\theta_{4\sigma},\theta_{5\sigma},\theta_{6\sigma},\theta_7\ldots)$ and $(\theta_{1\tau},\theta_{2\tau},\theta_{3\tau},\theta_{4\tau},\theta_{5\tau},\theta_{6\tau},\theta_7\ldots)$ are in distinct T^φ-orbits, thus these points are in distinct T-orbits, and so in distinct S-orbits. So in this case $\mathrm{Reg}(S,k-1) \geqslant |\mathrm{Sym}(\{1,2\}) \times \mathrm{Sym}(\{3,4\}) \times \mathrm{Sym}(\{5,6\})| = 8$.

Now assume that $6 \leqslant k \leqslant 8$. Denote by Ξ the subset $\{(\theta_{1^\sigma},\ldots,\theta_{(k-1)^\sigma}) \mid \sigma \in \mathrm{Sym}_{k-1}\}$ of Δ^{k-1}. Then T^φ acts on Ξ and every point of Ξ is T^φ-regular. Moreover $|\Xi| = |\mathrm{Sym}_{k-1}| = (k-1)!$. We also have that T^φ is a solvable subgroup of Sym_{k-1}. It is immediate, that $|T^\varphi| \leqslant 24$ for $k=6$, $|T^\varphi| \leqslant 72$ for $k=7$, and $|T^\varphi| \leqslant 144$ for $k=8$. Now the number of T^φ-orbits on Ξ is equal to $\frac{(k-1)!}{|T^\varphi|} \geqslant 5$. □

Now we outline some technical tool for searching the base size in almost simple groups. If G acts transitively on Ω, then given $x \in G$ by $\mathrm{fpr}(x)$ we denote the fixed point ratio of x, i.e. $\mathrm{fpr}(x) = |\mathrm{fix}(x)|/|\Omega|$, where $\mathrm{fix}(x) = \{\omega \in \Omega \mid \omega^x = \omega\}$. If G acts transitively and H is a point stabilizer, then

$$\mathrm{fpr}(x) = \frac{|x^G \cap H|}{|x^G|}. \tag{3}$$

Indeed, if $1 = g_1, g_2, \ldots, g_k$ is the right transversal, $\{H, H^{g_2}, \ldots, H^{g_k}\}$ is the set of point stabilizers. Now

$$|C_\Omega(x)| = |\{i \mid x \in H^{g_i}\}| = \frac{|\{g \mid x^{g^{-1}} \in H\}|}{|H|} = \frac{|x^G \cap H||C_G(x)|}{|H|}.$$

On the other hand $|\Omega| = k = |G:H| = \frac{|x^G||C_G(x)|}{|H|}$.

The base size can be bounded by using the following arguments. Assume that G acts faithfully and let $Q(G,c)$ denote the probability that arbitrary chosen element of Ω^c is not a G-regular point. Clearly, $b(G)$ is the minimal c such that $Q(G,c) < 1$. In particular, if $Q(G,c) < 1$ then $b(G) \leqslant c$. Clearly, an element of Ω^c is not a G-regular point if and only if some element x of prime order fixes it. Notice also that the probability for arbitrary chosen element of Ω^c to be stable under x is not greater than $\mathrm{fpr}(x)^c$. Denote by $\mathcal{P}$ the set of elements of G whose order is equal to a prime number. Let $x_1,\ldots,x_k$ be representatives of the conjugacy classes of elements from $\mathcal{P}$. Since G acts transitively, the formulae (3) shows that $\mathrm{fpr}(x)$ does not depend on the choice of the representative of a conjugacy class.

Thus the following chain of inequalities holds.

$$Q(G,c) \leqslant \sum_{x \in \mathcal{P}} \mathrm{fpr}(x)^c = \sum_{i=1}^{k} |x_i^G| \cdot \mathrm{fpr}(x_i)^c =: \widehat{Q}(G,c). \tag{4}$$

In particular, we can use the upper bound for $\mathrm{fpr}(x)$ in order to bound $\widehat{Q}(G,c)$ and so to bound $Q(G,c)$. The following lemma is the main technical tool for this bound.

Lemma 2. (T.Burness, M.Liebeck, A.Shalev, 2009)
Let G be a transitive group of permutations on Ω and H be a point stabilizer. Assume that $x_1,\ldots,x_k$ are representatives of distinct conjugacy classes such that the inequalities $\sum_i |x_i^G \cap H| \leqslant A$ and $|x_i^G| \geqslant B$ hold for all $i=1,\ldots,k$. Then the inequality $\sum_{i=1}^k |x_i^G|\cdot \mathrm{fpr}(x_i)^c \leqslant B(A/B)^c$ holds for every $c \in \mathbb{N}$.

Let G be an almost simple group with simple socle S. Consider a maximal subgroup H of G such that $S \nleqslant H$. The action of G on G/H is called standard, if

(a) $S \simeq \mathrm{Alt}_n$ and $H \cap S$ either is equal to either $\mathrm{Sym}_m \times \mathrm{Sym}_{n-m} \cap \mathrm{Alt}_n$ or $\mathrm{Sym}_m \wr \mathrm{Sym}\, n/m$;

(b) S is a classical group and $H \cap S$ is parabolic (lies in the first Aschbacher's class).

In the remaining cases the action is called nonstandard.

Theorem 12. (T.Burness, M.Liebeck, A.Shalev, 2009)
In the above notation assume that the action of G on G/H is nonstandard. Then $b_H(G) \leqslant 7$.

Theorem 13.
If S is a simple group of Lie type and $H = N_G(B)$, where B is a Borel subgroup of S, then $b_H(G) \leqslant 4$.

Problem 1. (KT, Question 17.41)
Let S be a maximal solvable subgroup of a finite group G.
(a) Is it true that $b_S(G) \leqslant 7$?
(b) Is it true that $b_S(G) \leqslant 5$?

Problem 2. (KT, Question 15.40)
Let N be a nilpotent subgroup of a finite simple group G. Does there exist $\sigma \in G$ such that $N \cap N^\sigma = \{\varepsilon\}$?

Problem 3. (KT, Question 17.40)
Let N be a nilpotent subgroup of a finite group G. Does there exist $\sigma, \tau \in G$ such that $N \cap N^\sigma \cap N^\tau \leqslant F(G)$.