Fusion systems VI: Connections to the classification of finite simple groups

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Today

Overview for today:

- Exercise 6 of Exercise sheet 2.
- The local theory of fusion systems: What we can, what we can't do in fusion systems.
- The classification of finite simple groups and a new proof via a classification of simple fusion systems. (Aschbacher's programme.)
- Intro: Partial groups, localities.

Hypothesis

Throughout, let p be a prime, and let $\mathcal{F}$ be a saturated fusion system on a p-group S.

- We can form normalizers and centralizers of subgroups of S, and p-local subsystems.

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- There are normal subgroups (i.e. subgroups of S normal in $\mathcal{F}$), normal subsystems, simple fusion systems, subnormal subsystems.

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- There are morphisms and factor systems modulo strongly closed subgroups.
- Warning: Kernels of morphisms don't correspond to normal subgroups or normal subsystems.
- There is a notion of a composition series and a Jordan–Hölder Theorem for fusion systems.
- There are centralizers of normal subsystems defined.
- Warning: There are no centralizers or normalizers of arbitrary subsystems defined.

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- There are definitions of $O^p(\mathcal{F})$ and $O^{p'}(\mathcal{F})$.
- There are definitions of quasisimple fusion systems, components of $\mathcal{F}$, $E(\mathcal{F})$ and the generalized Fitting subsystem $F^*(\mathcal{F}) = E(\mathcal{F})O_p(\mathcal{F})$.

Classification of simple fusion systems

Aschbacher suggests the following strategy:

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Example

It is easy to classify saturated 2-fusion systems on dihedral groups, but this doesn't tell us very much about finite groups with dihedral Sylow 2-subgroup (which are classified by Gorenstein and Walter).

Advantages

Fusion systems don't see normal p'-subgroups: If G is a finite group, $S \in Syl_p(G)$ and $\overline{G} := G/O_{p'}(G)$ then $\mathcal{F}_S(G) \cong \mathcal{F}_{\overline{S}}(\overline{G})$.

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In particular, classification results for simple fusion systems (at the prime 2) lead to a structured search for exotic fusion systems (at least at the prime 2). Hopefully, one will gain a better understanding why and when exotic fusion systems arise.

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- There is no established notion of an action of a fusion system. There is no "meaningful" representation theory of fusion systems.
- (The 2-fusion system of a simple group is not necessarily a simple fusion system.)

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If G is of Lie rank of 1 then $\mathcal{F}_S(G) = \mathcal{F}_S(N_G(S))$, so S is normal in $\mathcal{F}_S(G)$. Thus, $\mathcal{F}_S(S) \trianglelefteq \mathcal{F}_S(G)$ and $\mathcal{F}_S(G)$ is not simple.

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Let $G = A_n$ be the alternating group on n letters and let $S \in \mathrm{Syl}_p(G)$. Write $n = ap + b$ with $0 \leq b < p$. Then $\mathcal{F}_S(G)$ is simple if and only if $n \geq p^2$ and $b \in \{0, 1\}$.

Example

Let G be a sporadic group, $p = 2$ and $S \in \mathrm{Syl}_2(G)$. Then $\mathcal{F}_S(G)$ is simple unless $G = J_1$.

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If G is a finite simple group of Lie type in characteristic $r \neq p$, then to my knowledge there are only isolated results addressing the question whether $\mathcal{F}_S(G)$ is simple for $S \in \mathrm{Syl}_p(G)$.

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Aschbacher says he has analyzed the case of exceptional groups of Lie type. Ruiz analyzed the case that $G = PSL_n(r^m)$.

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$$C_M(O_p(M)) \leq O_p(M).$$

Theorem (Gorenstein–Walter Dichotomy Theorem)

Let G be a finite simple group of 2-rank at least 3. Then G is either of component type or of characteristic 2-type.

The Dichotomy Theorem for groups

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Modern approaches to the proof of the classification of finite simple groups use somewhat different case distinctions, and there is still some discussion about this. Common to all approaches is that one wants to treat more cases with characteristic 2-type methods.

Fusion systems of characteristic p-type and of component type

Definition

- The fusion system $\mathcal{F}$ is said to be of component type if there exists a an element $t \in S$ of order p such that $\langle t \rangle$ is fully centralized and $C_{\mathcal{F}}(t) = C_{\mathcal{F}}(\langle t \rangle)$ has a component.

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- $\mathcal{F}$ is said to be of characteristic p-type if every p-local subsystem (i.e. every normalizer of a non-trivial fully normalized subgroup of S) is constrained.

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The proof of the dichotomy theorem for fusion systems requires only a version of L-balance for fusion systems.

Groups and fusion systems of characteristic p-type and component type

Example

Let G be a finite group, $S \in Syl_p(G)$, K a component of G and $T := K \cap S$. Then $\mathcal{F}_T(K)$ is a component of $\mathcal{F}_S(G)$ if and only if the p-fusion system of $K/Z(K)$ is simple.

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Example

- *If G is a finite group of characteristic p-type then the p-fusion system of G is a fusion system of characteristic p-type. The converse is false.*
- *Assume there is an involution $t \in G$ and a component L of $C_G(t)/O_{2'}(C_G(t))$ such that the p-fusion system of $L/Z(L)$ is simple. Then the p-fusion system of G is of component type.*

Aschbacher's programme

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The complementary case for groups is treated in an ongoing programme of Meierfrankenfeld, Stellmacher and Stroth to understand groups of local characteristic p for any prime p, and to classify such groups if $p = 2$.

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In a classification of simple fusion systems of component type at the prime $p = 2$, one therefore usually assumes that the composition factors of p-local subsystems are either simple fusion systems which come from finite simple groups, or are Solomon-Benson fusion systems.

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Aschbacher's programme would avoid a proof of the B-conjecture completely.

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Setting $H := C_G(s)$, L is a component of $C_H(t)$. Using L-balance and the B-conjecture, we get $L \leq L(C_H(t)) \leq L(H) = E(H)$.

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The basic idea is now that L is either maximal in a suitable sense, or we can move on to a "larger" component of $H = C_G(s)$.

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(1) $L = D$.

(2) $L < D$ and $D^t = D$.

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Lemma

Let $t \in G$ be an involution and let L be a component of $C_G(t)$. Suppose $E(G) = L(G)$. Then there exists a component D of G such that one of the following three cases holds:

(1) $L = D$.

(2) $L < D$ and $D^t = D$.

(3) $D \neq D^t$ and $L = E(C_{DD^t}(t)) = \{dd^t : d \in D\}$ *is the homomorphic image of* D *under the map* $d \mapsto dd^t$.

Lemma

Let $s, t \in G$ be two commuting involutions. Let L be a component of $C_G(t)$ such that $[L, s] = 1$. Suppose $E(C_G(s)) = L(C_G(s))$.

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Pumping up

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Results similar to the previous two lemmas can be proved accordingly for fusion systems, and the formulations become easier.

Maximal components of involution centralizers

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Standard subsystems of fusion systems

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In his Copenhagen lecture notes, Aschbacher suggests a way of defining a standard subsystem, but using this notion one does not get a standard subsystems in all cases where one would want to get one.

It would thus be of interest to define the centralizer of a subsystem at least in special cases which allow for a more robust definition of standard subsystems.

Standard form problems

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Problems which should be standard form problems for fusion systems in any sensible definition have been treated by Lynd and Welz.

Aschbacher's classical involution theorem and Walter's theorem

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It remains to treat cases where a component in the centralizer of an involution is a Solomon–Benson fusion system. Justin Lynd and I are planning to work on this problem over the next year.

Fusion systems of characteristic p-type

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We will next introduce partial groups and localities (as defined by Andrew Chermak). Working with localities gives an elegant way of fitting these models for p-local subsystems together.

Localities

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Work in progress of Andrew Chermak and myself shows that normal subsystems of fusion systems correspond to kernels of homomorphisms of certain localities.

This implies that there is something like the product of two normal subsystems in a saturated fusion system.

Setup

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For $u, v \in \mathbf{W}(\mathcal{L})$ let $u \circ v$ be the concatenation of u and v.

Write $\emptyset$ for the empty word.

Partial groups

Definition

A partial group is a set $\mathcal{L}$ together with a subset $\mathbf{D} \subseteq \mathbf{W}(\mathcal{L})$, a map

$$\Pi \colon \mathbf{D} \to \mathcal{L}$$

(called the partial product) and an involutory bijection $\mathcal{L} \to \mathcal{L}, f \mapsto f^{-1}$ (called the inversion map).

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(4) $w \in \mathbf{D} \Longrightarrow w^{-1} \circ w \in \mathbf{D}$ and $\Pi(w^{-1} \circ w) = \mathbf{1}$, where $\mathbf{1} := \Pi(\emptyset)$ and $w^{-1} = (f_n^{-1}, \ldots, f_1^{-1})$ for $w = (f_1, \ldots, f_n)$.

The end

Thank you!!!