1. Lecture 2: The O'Nan-Scott theorem

The O'Nan-Scott theorem essentially provides an identification and a description of several types of finite primitive groups: for each type, we have additional information about either the abstract group theoretical structure, or the nature of the action, or both.

The ordering, the labeling and the amount of subdivision of the types depends on the taste, the requirements of the applications: for some applications a very coarse subdivision is enough, for other applications a finer subdivision is necessary.

Let G be a group. The ***socle*** of G is the subgroup generated by the minimal normal subgroups of G. A permutation group G on Ω is said to be ***regular*** if G is both transitive and semiregular on Ω, that is, the only element of G fixing some element of Ω is the identity permutation.

For the exposition I have chosen a subdivision of the finite primitive groups into eight types. This subdivision is finer than the subdivision proved in the original papers and follows a suggestion of Laszlo Kovacs. The O'Nan-Scott theorem shows that every finite primitive permutation group belongs to one and only one of these eight types.

Let G be a finite primitive permutation group on Ω and let $\omega \in \Omega$. The stabilizer of ω is the subgroup $G_\omega := \{g \in G \mid \omega^g = \omega\}$.

Now, G has at most two minimal normal subgroups and, if G does have two minimal normal subgroups, then each is the centralizer of the other and they are isomorphic.

In particular, assume that G has a unique minimal normal subgroup N and that this subgroup N happens to be abelian. Then N is the unique minimal normal subgroup of G and is the socle of G. The minimality yields N is an elementary abelian p-group of order p^d, for some prime number p and some positive integer d. Since N is transitive and abelian, we get that N is regular on Ω and G is the semidirect product $N \rtimes G_\omega$. Thus we may identify Ω with N and, in turn, N with the d-dimensional vector space $\mathbb{F}_p^d$, where $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ is the finite field with p elements. Under this identification, by taking ω as the zero vector, we get that G_ω is an irreducible subgroup of non-singular linear transformations on $N = \mathbb{F}_{p^d}$. This is the first type: HA Holomorph Abelian, because G is contained in the holomorph of N. (The holomorph of a group X is the semidirect product $X \rtimes \mathrm{Aut}(X)$, where $\mathrm{Aut}(X)$ acts naturally on X.)

HA: $\Omega = \mathbb{F}_p^d$, for some prime number p and some positive integer d and G is the semidirect product $N \rtimes H$ and is a subgroup of the affine general linear group $\mathrm{AGL}_d(p)$, where N is the group of translations and H is an irreducible subgroup of $\mathrm{GL}_d(p)$.

Broadly speaking, a complete understanding of the primitive groups of HA type requires a complete understanding of all irreducible faithful representations of finite groups.

Exercise 1. Prove that if G is quasiprimitive and its socle is abelian, then G is primitive.

Exercise 2. Prove that if $G = N \rtimes G_\omega$ is primitive of affine type and is basic, then G_ω acts primitively as a linear group on N, that is, G_ω leaves invariant no direct sum decomposition of N.

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Exercise 4. Prove that if $G = N \rtimes G_\omega$ is primitive of affine type, then G_ω acts faithfully on each of its orbits. Find a finite primitive group with G_ω not acting faithfully on one of its orbits.

In all the other O'Nan-Scott types, each minimal normal subgroup of G is non-abelian. Let N be one of these minimal normal subgroups. Then $N = T_1 \times T_2 \times \cdots \times T_d$ for some positive integer d and $T_i \cong T$, for each $i \in \{1, \ldots, d\}$, for a fixed non-abelian finite simple group. Suppose then that G has another minimal normal subgroup (different from N), say M. Then $M \cong N$, M and N are both regular on Ω and $\mathrm{Soc}(G) = N \times M$. In this case, G is contained in the holomorph $N \rtimes \mathrm{Aut}(N)$ of N: under this embedding $M = \{\iota_n^{-1} n \mid n \in N\}$, where $\iota_n : N \to N$ is the inner automorphism induced by $n \in N$. In our O'Nan-Scott subdivision, this case splits into two subcases: $d = 1$ and $d > 1$. When $d = 1$, G is said to have O'Nan-Scott type HS (Holomorph Simple group). When $d > 1$, G is said to have O'Nan-Scott type HC (Holomorph Compound, for a reason that should be clearer later). Let us denote by $\mathrm{Inn}(N) := \{\iota_n \mid n \in N\}$ the group of inner automorphisms of N. Moreover, given $\varphi \in \mathrm{Aut}(N)$ and $n \in N$, we use an exponential notation and we denote n^φ the image of n under the mapping φ.

HS: $\Omega = T$, for some non-abelian simple group T and $T \rtimes \mathrm{Inn}(T) \leq G \leq \mathrm{Hol}(T) = T \rtimes \mathrm{Aut}(T)$. Given $\omega \in \Omega$, $t \in T$ and $\varphi \in \mathrm{Aut}(T)$, we have $t\varphi : \omega \mapsto \omega^\varphi t^\varphi$.

HC: $\Omega = T^d = N$, for some non-abelian simple group T and for some positive integer d, and $N \rtimes \mathrm{Inn}(N) \leq G \leq \mathrm{Hol}(N) = N \rtimes \mathrm{Aut}(T)$. Given $\omega \in \Omega$, $n \in N$ and $\varphi \in \mathrm{Aut}(N)$, we have $n\varphi : \omega \mapsto \omega^\varphi n^\varphi$. Moreover, if ω is the identity element of N, then $\mathrm{Inn}(N) \leq G_\omega \leq \mathrm{Aut}(N)$ and G_ω acts primitively by conjugation on the d simple direct factors $\{T_1, \ldots, T_d\}$ of N. (The hypothesis on G_ω is needed to guarantee that the action is primitive.)

Broadly speaking, a complete understanding of finite primitive groups of HS and HC type requires a complete understanding of non-abelian simple groups.

For the remaining types, N is the unique minimal normal subgroup of G and $\mathrm{Soc}(G) = N$. When $d = 1$, that is, $N = T$ is a non-abelian simple group, we have $T \leq G \leq \mathrm{Aut}(T)$ and hence G is an almost simple group with socle T. This case is named AS: Almost Simple.

AS: $T \leq G \leq \mathrm{Aut}(G)$, for some non-abelian simple group T. Given $\omega \in \Omega$, $G = TG_\omega$, $T \nleq G_\omega$ and G_ω is a maximal subgroups of G.

Observe that in this type we have a concrete abstract description of G, but no additional information on the action. Broadly speaking, a complete understanding of finite primitive groups of AS type requires a classification of the maximal subgroups of the almost simple groups with socle a non-abelian simple groups.

Exercise 5. In every primitive group G of AS type on Ω, the socle T acts transitively, but not necessarily primitively on Ω. Consider the projective space $\mathrm{PG}(3, 2)$, that is, the Fano plane and Ω consisting of the pairs of collinear point-line pairs: $\Omega := \{\{p, \ell\} \mid p \text{ point}, \ell \text{ line}, p \in \ell\}$. Let G be the group generated by $\mathrm{PSL}_3(2) = \mathrm{PGL}_3(2)$ and by the polarity ι swapping points and lines in the Fano plane. Verify that G acts primitively on Ω. However, given a fixed point p, the set $\{\{p, \ell\} \mid p \in \ell\}$ is a block of imprimitivity for the action of $\mathrm{PSL}_3(2)$ on Ω.

The description of the primitive groups of SD (Simple Diagonal type) is a little more laborious. Let $\ell \geq 1$ and let T be a non-abelian simple group. Consider the group $N = T^{\ell+1}$ and $D = \{(t, \ldots, t) \in N \mid t \in T\}$, a diagonal subgroup of N. Set $\Omega := N/D$, the set of right cosets of D in N. Then $|\Omega| = |T|^{\ell}$. Moreover we may identify each element $\omega \in \Omega$ with an element of T^{ℓ} as follows: the right coset $\omega = D(\alpha_0, \alpha_1, \ldots, \alpha_\ell)$ contains a unique element whose first coordinate is 1, namely, the element $(1, \alpha_0^{-1}\alpha_1, \ldots, \alpha_0^{-1}\alpha_\ell)$. We choose this distinguished coset representative and we denote the element $D(1, \alpha_1, \ldots, \alpha_\ell)$ of Ω simply by

$$[1, \alpha_1, \ldots, \alpha_\ell].$$

Now the element φ of $\mathrm{Aut}(T)$ acts on Ω by

$$[1, \alpha_1, \ldots, \alpha_\ell]^{\varphi} = [1, \alpha_1^{\varphi}, \ldots, \alpha_\ell^{\varphi}].$$

Note that this action is well-defined because D is $\mathrm{Aut}(T)$-invariant. Next, the element $(t_0, \ldots, t_\ell)$ of N acts on Ω by

$$[1, \alpha_1, \ldots, \alpha_\ell]^{(t_0, \ldots, t_\ell)} = [t_0, \alpha_1 t_1, \ldots, \alpha_\ell t_\ell] = [1, t_0^{-1}\alpha_1 t_1, \ldots, t_0^{-1}\alpha_\ell t_\ell].$$

Observe that the action induced by $(t, \ldots, t) \in N$ on Ω is the same as the action induced by the inner automorphism corresponding to conjugation by t. Finally, the element σ in $\mathrm{Sym}(\{0, \ldots, \ell\})$ acts on Ω simply by permuting the coordinates. Note that this action is well-defined because D is $\mathrm{Sym}(\ell+1)$-invariant.

The set of all permutations we described generates a group W isomorphic to $T^{\ell+1} \cdot (\mathrm{Out}(T) \times \mathrm{Sym}(\ell+1))$. A subgroup G of W containing the socle N is primitive if either $\ell = 1$ or G acts primitively by conjugation on the $\ell + 1$ simple direct factors of N. When $\ell = 1$, and G does not act primitive by conjugation on $\{T_1, T_2\}$ (the simple direct factors of N), we get that G is actually a subgroup of $T^2.\mathrm{Out}(T) = T \rtimes \mathrm{Aut}(T)$ and hence this case has been discussed already and G is a primitive group of HS type.

When we exclude this case, the remaining groups are called the primitive groups of Diagonal type. Write

$$M = \{(t_0, t_1, \ldots, t_\ell) \in N \mid t_0 = 1\}.$$

Clearly, M is a normal subgroup of N acting regularly on Ω. Since the stabilizer in W of the point $[1, \ldots, 1]$ is $\mathrm{Sym}(\ell+1) \times \mathrm{Aut}(T)$, we obtain that

$$W = (\mathrm{Sym}(\ell+1) \times \mathrm{Aut}(T))M.$$

Moreover, every element $x \in W$ can be written uniquely as $x = \sigma\varphi m$, with $\sigma \in \mathrm{Sym}(\ell+1)$, $\varphi \in \mathrm{Aut}(T)$ and $m \in M$.

SD: $\Omega = T^{\ell} = N$, for some non-abelian simple group T and for some positive integer ℓ, and $N = T^{\ell+1} = \mathrm{Soc}(W) \leq G \leq W$, with the action described above.

We now describe the primitive groups of Compound Diagonal type CD: these use the primitive wreath product construction.

CD: $\Omega = \Delta^d$ and $N = T^k \leq G \leq H \,\mathrm{wr}\, \mathrm{Sym}(k)$ for some divisor d of k, where $d \geq 2$ and $k/d \geq 2$, T is a non-abelian simple group, $H \leq \mathrm{Sym}(\Delta)$, $\mathrm{Soc}(H) = T^{k/d}$ and H is primitive on Δ of O'Nan-Scott type SD. Moreover, the action of G on the simple direct factors of N is transitive.

Therefore for primitive groups of CD type the subgroup H induces on Δ a primitive group of SD type. We now turn to primitive groups of TW type: Twisted Wreath. This follows a construction due to Michio Suzuki. Let T be a non-abelian simple group, H be a group, L be a subgroup of H and $\phi : L \to \mathrm{Aut}(T)$ be a homomorphism with the image of ϕ containing the inner automorphisms of T. Let R be a set of left coset representatives of L in H and T^H be the set of all functions $f : H \to T$ from H to T. Clearly, T^H is a group under pointwise multiplication, and H acts as a group of automorphisms on T^H by setting $f^x(z) = f(xz)$, for $f \in T^H$ and for $x, z \in H$. Write

$$N = \{f \in T^H \mid f(zl) = f(z)^{\phi(l)} \text{ for all } z \in H \text{ and } l \in L\}.$$

It is easy to verify that N is an H-invariant subgroup of T^H isomorphic to T^R. In fact, the restriction mapping $f \mapsto f_{|R}$ is an isomorphism of N onto T^R. The semidirect product $G = N \rtimes H$ is said to be the twisted wreath product determined by H and ϕ. The group G acts on $\Omega = N$ by setting $\omega^{nh} = (\omega n)^h$, for each $\omega \in \Omega$, $n \in N$ and $h \in H$. (In particular, N acts on Ω by right multiplication and H acts on Ω by conjugation.) It is rather intricate to describe when this action is primitive. However, it is possible to prove that, if H is a primitive permutation group with point stabilizer L and if the image of ϕ is not a homomorphic image of H, then G acts primitively on Ω.

TW: $\Omega = T^k = N$, for some non-abelian simple group T and for some positive integer $k \geq 6$, and G is a twisted wreath product as defined above and with the action described above.

The condition on $k \geq 6$ is necessary to ensure that the action of G is primitive (the proof that $k \geq 6$ depends upon Schreier's conjecture and hence ultimately on the CFSGs). Observe that if G is primitive of TW type, then it is contained in the Holomorph of its socle. However, this case is different from the HC case because G does have a unique minimal normal subgroup, and primitive groups of HC type do have two minimal normal subgroups.

Example 1.1. Consider $T = L = \mathrm{Alt}(5)$, $H = \mathrm{Alt}(6)$ and $\phi : L \to \mathrm{Aut}(T)$ the identity mapping, where we are identifying T with the inner automorphisms of T. Applying the construction above, we obtain a primitive group $G = \mathrm{Alt}(5)^6 \rtimes \mathrm{Alt}(6)$ acting on $\Omega = \mathrm{Alt}(5)^6$ which is primitive of TW type: indeed, the image $\mathrm{Alt}(5) = T$ of L is not an homomorphic image of $H = \mathrm{Alt}(6)$.

The last type is PA: Product Action.

PA: $\Omega = \Delta^d$, $N = T^d \leq G \leq H \,\mathrm{wr}\, \mathrm{Sym}(d)$, for some non-abelian simple group T and for some positive integer $d \geq 2$. Moreover, the group H is primitive of AS type with socle T and the action of G is the primitive product action on the Cartesian product Δ^d. Moreover, the action of G on the simple direct factors of N is transitive.