The arithmetics of binary quadratic operads

Algebras and related groups

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Plan of the mini-course

1. Quadratic associative algebras: Koszul duality and Manin products
2. Examples of computing & Exercises
3. Varieties of non-associative algebras: linearization of identities
4. Free algebras and their multi-linear parts; Examples
5. Operads: from trees to varieties
6. Binary quadratic operads, Koszul duality
7. Manin white products of operads
8. Manin black product of operads

1. Quadratic associative algebras

[Loday, Vallette: Algebraic operads]

[Polishchuk, Positselski: Quadratic algebras]

$$\left\{\begin{array}{ll} V & \text{vector space over a field } \Bbbk \\ R \subseteq V \otimes V & \text{subspace} \end{array}\right.$$

$\dim V < \infty$

quadratic data

$\Downarrow$

tensor algebra $= \Bbbk \oplus V \oplus V^{\otimes 2} \oplus V^{\otimes 3} \oplus \dots$

Def

$$\mathcal{A}(V,R) = T(V)/(R)$$

ideal in $T(V)$ generated by R

quadratic associative algebra

defined by the quadratic data (V,R)

In other words:

Let $x_1, \dots, x_n$ be a basis of V

Then $$T(V) \cong \Bbbk\langle x_1, x_2, \dots, x_n \rangle$$

non-commutative polynomials in the variables $x_1, \dots, x_n$

$$R \subseteq V \otimes V = \left\{ \sum_{i,j=1}^{n} \alpha_{ij}\, x_i x_j \,\middle|\, \alpha_{ij} \in \Bbbk \right\}$$

homogeneous non-commutative polynomials of $\deg = 2$

Example 1. $\dim V = 2$, x_1, x_2 basis

$$R = \operatorname{span}\{x_1\otimes x_1 + x_2\otimes x_2,\ x_1\otimes x_2 - x_2\otimes x_1\}$$

$$A = \mathcal{A}(V,R) = \Bbbk\langle x_1, x_2 \mid x_1^2 + x_2^2 = 0,\ x_1x_2 = x_2x_1\rangle$$

commutative

If $\sqrt{-1}\in\Bbbk$ then

$$x_1^2 + x_2^2 = (x_1 - ix_2)(x_1 + ix_2)$$

so $A \cong \Bbbk[u,v]/(uv)$

Example 2. $\dim U = 2$,

y_1, y_2 basis

$$S = \operatorname{span}\{y_1\otimes y_2 + y_2\otimes y_1,\ y_1\otimes y_1,\ y_2\otimes y_2\}$$

$$B = \mathcal{A}(U,S) = \Bbbk\langle y_1, y_2 \mid y_1^2 = y_2^2 = y_1y_2 + y_2y_1 = 0\rangle \cong \Lambda_2$$

the exterior algebra of U

Suppose (V, R) is a quadratic data, $R \subseteq V \otimes V$

$V^* = \mathrm{Hom}(V, \Bbbk)$ dual space.

Then $(V \otimes V)^* \cong V^* \otimes V^*$ natural isomorphism:

$\langle \xi \otimes \eta, u \otimes v \rangle = \langle \xi, u \rangle \langle \eta \otimes v \rangle$

for $\xi, \eta \in V^*$, $u, v \in V$

thus

$R^\perp \subseteq (V \otimes V)^*$

may be considered as a subspace of $V^* \otimes V^*$

$\Downarrow$

$(V^*, R^\perp)$ is again a quadratic data

For $A = \mathcal{A}(V, R)$, $\mathcal{A}(V^*, R^\perp) =: A^!$ $\left\{ \begin{array}{l} \text{Koszul dual} \\ \text{quadratic algebra} \end{array} \right.$

Example (see Example 1 on page 5)

Let $A = \Bbbk\langle x_1, x_2 \mid x_1^2 + x_2^2 = 0,\ x_1x_2 - x_2x_1 = 0 \rangle = \mathcal{A}(V, R)$

x_1, x_2 basis of V

$V \otimes V$: $\underset{\underset{e_1}{\parallel}}{x_1^2}$, $\underset{\underset{e_2}{\parallel}}{x_1x_2}$, $\underset{\underset{e_3}{\parallel}}{x_2x_1}$, $\underset{\underset{e_4}{\parallel}}{x_2^2}$ basis

$$\left.\begin{array}{l} V^* \text{ has dual basis} \\ \xi_1 = x_1^*,\quad \xi_2 = x_2^* \end{array}\right\} \Rightarrow \begin{array}{ll} \xi_1^2 = e_1^* & \xi_1\xi_2 = e_2^* \\ \xi_2\xi_1 = e_3^* & \xi_2^2 = e_4^* \end{array}$$

$$R^\perp = \left(\operatorname{span}\{e_1 + e_4,\ e_2 - e_3\}\right)^\perp = \operatorname{span}\{\underset{\xi_1^2 - \xi_2^2}{\underbrace{e_1^* - e_4^*}},\ \underset{\xi_1\xi_2 + \xi_2\xi_1}{\underbrace{e_2^* + e_3^*}}\}$$

$$\boxed{A^! = \Bbbk\langle \xi_1, \xi_2 \mid \xi_1^2 = \xi_2^2,\ \xi_1\xi_2 = -\xi_2\xi_1 \rangle}$$

HW (Exercise 1): Calculate $B^!$ for

Example 2 on page 5

$$B = \Bbbk\langle y_1, y_2 \mid y_1^2 = y_2^2 = y_1y_2 + y_2y_1 = 0\rangle$$

Proposition Let $\Lambda_n = \Bbbk\langle y_1, \dots, y_n \mid y_i^2 = 0,\ y_iy_j + y_jy_i = 0\rangle$
$i, j = 1, \dots, n$

Then

$$\Lambda_n^! = \Bbbk[\eta_1, \dots, \eta_n]$$

commutative polynomial algebra in $\eta_1, \dots, \eta_n$

Pf

Compare the dimensions to prove $\eta_i\eta_j - \eta_j\eta_i$ span $S^\perp$

Proposition

For every quadratic associative $A = \mathcal{A}(V, R)$ we have $(A^!)^! \cong A$

$\dim V < \infty$

Pf $(U^*)^* \cong U$, $(R^\perp)^\perp \cong R$

As a corollary, $\Bbbk[x_1, \dots, x_n]^! = \Lambda_n$

$\mathbb{S}(V)$ symmetric algebra

$\mathbb{E}(V)$ exterior algebra

Koszul dual

Let

$$\left.\begin{array}{l}(V, R) \\ \\ (U, S)\end{array}\right\} \text{ quadratic data} \qquad R \subseteq V \otimes V \qquad S \subseteq U \otimes U$$

$$A = \mathcal{A}(V, R) \qquad B = \mathcal{A}(U, S) \qquad \text{quadratic algebras}$$

Then

$$\begin{tikzcd} R \otimes U \otimes U \arrow[d] & \subseteq & V \otimes V \otimes U \otimes U \arrow[d, "(23)"] \\ (R \otimes U \otimes U)^{(23)} & \subseteq & (V \otimes U) \otimes (V \otimes U) \end{tikzcd}$$

Similarly, $(V \otimes V \otimes S)^{(23)} \subseteq (V \otimes U) \otimes (V \otimes U)$

$$\left.\begin{array}{c} \text{Hence,}\quad V\otimes U \\ \text{and} \\ R\circ S = \left(R\otimes U^{\otimes 2} + V^{\otimes 2}\otimes S\right)^{(23)} \end{array}\right\} \text{quadratic data}$$

$$A\circ B = \mathcal{A}\left(V\otimes U,\ R\circ S\right)$$

Manin white product of quadratic algebras A, B

Namely, $R\circ S$ = all relations that hold for $V\otimes U$ in $A\otimes B$

$$\mathbin{/\!/}$$

$A\circ B$ ← associative algebra, not quadratic in general

Properties

1) $A \circ B \cong B \circ A$

2) $(A \circ B) \circ C \cong A \circ (B \circ C)$

3) $\mathbb{k}[x] \circ B \cong B$

$\dim V = 1$

$R = 0 \implies R \circ S = \left(V^{\otimes 2} \otimes S \right)^{(23)} \cong S$

$$\left.\begin{array}{l}(V, R) \\ \\ (U, S)\end{array}\right\}\ \text{quadratic data} \qquad \begin{array}{l} R \subseteq V \otimes V \\ \\ S \subseteq U \otimes U \end{array}$$

$$A = \mathcal{A}(V, R) \qquad B = \mathcal{A}(U, S) \qquad \text{quadratic algebras}$$

Then $\left.\begin{array}{l} V \otimes U \\ \\ (R \otimes S)^{(23)} \end{array}\right\}$ quadratic data

$$A \bullet B = \mathcal{A}\left(V \otimes U,\ (R \otimes S)^{(23)}\right)$$

Manin black product of quadratic algebras A, B

Example

Calculate Manin product $A \circ B$ for

$$A = \Bbbk\langle x_1, x_2 \mid x_1^2 + x_2^2,\ x_1x_2 - x_2x_1\rangle$$

$$B = \Bbbk\langle y_1, y_2 \mid y_1^2,\ y_2^2,\ y_1y_2 + y_2y_1\rangle$$

Examples 1, 2 on page 5

$$R = \operatorname{span}\{x_1^2 + x_2^2,\ x_1x_2 - x_2x_1\}$$

$$S = \operatorname{span}\{y_1^2,\ y_2^2,\ y_1y_2 + y_2y_1\}$$

$z_{ij} = x_i \otimes y_j$, $i, j = 1, 2$

Method 1 (straight)

$$R \circ S = \operatorname{span}\{(x_1^2 + x_2^2) \otimes y_iy_j,\ (x_1x_2 - x_2x_1) \otimes y_iy_j,$$
$$x_ix_j \otimes y_1^2,\ x_ix_j \otimes y_2^2,\ x_ix_j \otimes (y_1y_2 + y_2y_1)\}$$

subspace in 16-dim space

Apply (23)

$$(x_1^2 + x_2^2) \otimes y_iy_j \rightsquigarrow x_1x_1 \otimes y_iy_j + x_2x_2 \otimes y_iy_j = z_{1i}z_{1j} + z_{2i}z_{2j} \quad \text{etc.}$$

$$R \circ S = \mathrm{span}\{ z_{1i} z_{1j} + z_{2i} z_{2j},\ z_{1i} z_{2j} - z_{2i} z_{1j},$$
$$z_{i1} z_{j1},\ z_{i2} z_{j2},\ z_{i1} z_{j2} + z_{i2} z_{j1} \}$$

20 vectors in 16-dim space

https://www.singular.uni-kl.de/

```
int n=2;
ring r=0,(a(1..n)(1..n)(1..n)(1..n)),lp;

int i; int j;

ideal T;

for (i=1; i<=n; i++)
  {
  for (j=1; j<=n; j++)
  {
  T=T,a(1)(i)(1)(j)+a(2)(i)(2)(j),
     a(1)(i)(1)(j)-a(2)(i)(1)(j),
     a(i)(1)(j)(1),
     a(i)(2)(j)(2),
     a(i)(1)(j)(2)+a(i)(2)(j)(1);
  }}
```

$a(i)(j)(k)(l) = z_{ij} z_{kl}$

$\Longrightarrow$

```
> std(T);
_[1]=a(2)(2)(2)(2)
_[2]=a(2)(2)(1)(2)
_[3]=a(2)(2)(1)(1)+a(2)(2)(2)(1)
_[4]=a(2)(1)(2)(2)+a(2)(2)(2)(1)
_[5]=a(2)(1)(2)(1)
_[6]=a(2)(1)(1)(2)+a(2)(2)(1)(1)
_[7]=a(2)(1)(1)(1)
_[8]=a(1)(2)(2)(2)
_[9]=a(1)(2)(1)(2)-a(2)(2)(1)(2)
_[10]=a(1)(2)(1)(1)-a(2)(2)(1)(1)
_[11]=a(1)(1)(2)(2)+a(1)(2)(2)(1)
_[12]=a(1)(1)(2)(1)
_[13]=a(1)(1)(1)(2)-a(2)(1)(1)(2)
_[14]=a(1)(1)(1)(1)
```

$$A \circ B = \mathbb{k}\langle z_{ij} \mid z_{22}^2,\ z_{22} z_{12},\ z_{22} z_{11} + z_{22} z_{21}, \ \ldots$$

$i,j = 1,2$

$\dim R \circ S = 14$

Method 2

Consider a generic $f = \sum_{i,j,k,l} \alpha_{ij}^{kl} z_{ij} z_{kl} \in (V \otimes U)^{\otimes 2}$

Expand $z_{ij} = x_i \otimes y_j$, $z_{kl} = x_k \otimes y_l$

and apply (23):

$$f^{(23)} = \sum_{i,j,k,l} \alpha_{ij}^{kl} \, x_i x_k \otimes y_j y_l$$

Reduce $y_j y_l$ mod S $\rightsquigarrow$ $y_1 y_2$ only

$y_i^2 = 0$, $y_2 y_1 = -y_1 y_2$

$$f^{(23)} \equiv \sum_{i,k} \alpha_{i1}^{k2} \, x_i x_k \otimes y_1 y_2 - \sum_{i,k} \alpha_{i2}^{k1} \, x_i x_k \otimes y_1 y_2$$

mod $V^{\otimes 2} \otimes S$

Reduce the result mod R

$$x_2^2 = -x_1^2$$

$$x_2 x_1 = x_1 x_2$$

$$\left.\begin{aligned} \equiv\ & \left(\alpha_{11}^{12} - \alpha_{21}^{22} - \alpha_{12}^{11} + \alpha_{22}^{21}\right) x_1^2 \otimes y_1 y_2 \\ + & \left(\alpha_{11}^{22} + \alpha_{21}^{12} - \alpha_{12}^{21} - \alpha_{22}^{11}\right) x_1 x_2 \otimes y_1 y_2 \end{aligned}\right\}$$

must be $= 0$ in $A \otimes B$

$$\Downarrow$$

$$f = \sum_{i,j,k,l} \alpha_{ij}^{kl} z_{ij} z_{kl} \in ROS \quad \text{iff} \quad \begin{cases} \alpha_{11}^{12} - \alpha_{21}^{22} - \alpha_{12}^{11} + \alpha_{22}^{21} = 0 \\ \alpha_{11}^{22} + \alpha_{21}^{12} - \alpha_{12}^{21} - \alpha_{22}^{11} = 0 \end{cases}$$

system of linear equations

2 eqs on 16 variables

Fundamental solutions $\Rightarrow$ basis of ROS $\Rightarrow \dim ROS = 14$

Example Calculate Manin product $A \bullet B$ for

$$A = \Bbbk\langle x_1, x_2 \mid x_1^2 + x_2^2,\ x_1x_2 - x_2x_1 \rangle$$
$$B = \Bbbk\langle y_1, y_2 \mid y_1^2,\ y_2^2,\ y_1y_2 + y_2y_1 \rangle$$

Examples 1, 2 on page 5

$$R = \mathrm{span}\{x_1^2 + x_2^2,\ x_1x_2 - x_2x_1\}$$
$$S = \mathrm{span}\{y_1^2,\ y_2^2,\ y_1y_2 + y_2y_1\}$$

$z_{ij} = x_i \otimes y_j$, $i,j = 1,2$

Method 1 (straight)

$$R \otimes S = \mathrm{span}\{(x_1^2 + x_2^2) \otimes y_i^2,\ (x_1^2 + x_2^2) \otimes (y_1y_2 + y_2y_1),$$
$$(x_1x_2 - x_2x_1) \otimes y_i^2,\ (x_1x_2 - x_2x_1) \otimes (y_1y_2 + y_2y_1)\} \quad (23)$$

6 vectors in 16-dim space

$$A \bullet B = \Bbbk\langle z_{ij} \mid z_{1i}^2 + z_{2i}^2,\ z_{11}z_{12} + z_{12}z_{11} + z_{21}z_{22},\ z_{1i}z_{2i} - z_{2i}z_{1i},$$
$$z_{11}z_{22} - z_{21}z_{12} + z_{12}z_{21} - z_{22}z_{11} \rangle$$

Method 2

see p.7 $A^! = \Bbbk\langle \xi_1, \xi_2 \mid \xi_1^2 - \xi_2^2,\ \xi_1\xi_2 + \xi_2\xi_1 \rangle$

Consider $\Psi: \Bbbk\langle y_1, y_2\rangle \longrightarrow A^! \otimes \Bbbk\langle z_{ij} \mid i,j = 1,2\rangle$

$$y_j \longmapsto \sum_{i=1}^{2} \xi_i \otimes z_{ij}$$

Find the image of S:

$$\Psi(y_j^2) = (\xi_1 \otimes z_{1j} + \xi_2 \otimes z_{2j})^2 = \xi_1^2 \otimes z_{1j}^2 + \xi_1\xi_2 \otimes z_{1j}z_{2j} + \xi_2\xi_1 \otimes z_{2j}z_{1j} + \xi_2^2 \otimes z_{2j}^2$$

$$= \xi_1^2 (z_{1j}^2 + z_{2j}^2) + \xi_1\xi_2 \otimes (z_{1j}z_{2j} - z_{2j}z_{1j})$$

basis of $A^!$

coefficients — relations in $A \bullet B$

$$\Psi(y_1 y_2 + y_2 y_1) = (\xi_1 \otimes z_{11} + \xi_2 \otimes z_{21})(\xi_1 \otimes z_{12} + \xi_2 \otimes z_{22})$$

$$+ (\xi_1 \otimes z_{12} + \xi_2 \otimes z_{22})(\xi_1 \otimes z_{11} + \xi_2 \otimes z_{21})$$

$$= \xi_1^2 \otimes (z_{11} z_{12} + z_{21} z_{22} + z_{12} z_{11} + z_{22} z_2)$$

$$+ \xi_1 \xi_2 \otimes (z_{11} z_{22} - z_{21} z_{12} + z_{12} z_{21} - z_{22} z_{11})$$

As a result:

$(R \otimes S)^{(23)}$ is spaned by all coefficients of $\Psi(S)$

$$z_{1j}^2 + z_{2j}^2$$

$$z_{1j} z_{2j} - z_{2j} z_{1j}$$

$$z_{11} z_{12} + z_{21} z_{22} + z_{12} z_{11} + z_{22} z_2$$

$$z_{11} z_{22} - z_{21} z_{12} + z_{12} z_{21} - z_{22} z_{11}$$

Why this method works:

$$A = \mathcal{A}(V, R) \qquad B = \mathcal{A}(U, S) \qquad \text{in general}$$

(x_i) basis of V $\qquad$ (y_j) basis of U

Restrict: $\Psi : U^{\otimes 2} \to \left(V^* \otimes V^* / R^\perp \right) \otimes (V \otimes U)^{\otimes 2}$

$$T(U) \supseteq U^{\otimes 2}, \qquad A^! \supseteq V^* \otimes V^* / R^\perp, \qquad T(V \otimes U) \supseteq (V \otimes U)^{\otimes 2}$$

$z_{ij} = x_i \otimes y_j$

$\xi_i = x_i^*$

$$\Psi(y_j y_k) = \Psi(y_j)\Psi(y_k) = \left(\sum_i \xi_i \otimes z_{ij} \right)\left(\sum_\ell \xi_\ell \otimes z_{\ell k} \right)$$

$$= \sum_{i,\ell} \xi_i \xi_\ell \otimes z_{ij} z_{\ell k}$$

$$R \hookrightarrow V \otimes V$$

dual map

$$R^* \twoheadleftarrow (V \otimes V)^*$$

onto

$\ker = R^\perp$

$$\Big\} \Rightarrow V^* \otimes V^* / R^\perp \cong R^*$$

i.e., $\Psi(p) \in R^* \otimes (V \otimes U)^{\otimes 2}$ for $p \in U^{\otimes 2}$

$\forall f \in R$: $\boxed{\left(\langle f, \cdot \rangle \otimes id\right) \Psi(p) \overset{(23)}{=} f \otimes p}$

straightforward check for

$$f = \sum d_{ij} x_i x_j$$

Coefficients of $\Psi(S) \Leftrightarrow R \otimes S$

(23) as desired

Exercise

1. Calculate $A^! \circ B^!$ and $A^! \bullet B^!$ for A, B as above

2. Compare the results with

$$(A \bullet B)^! \quad \text{and} \quad (A \circ B)^!$$

Proposition

Let A and B be quadratic associative algebras.

Then $$(A \circ B)^! = A^! \bullet B^!$$

Pf Elementary linear algebra: if $R \subseteq X$, $S \subseteq Y$

then $$(X \otimes S + R \otimes Y)^\perp = (R^\perp \otimes S^\perp) \quad \text{in } X \otimes Y$$