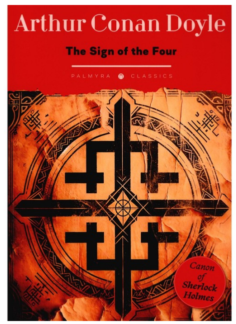
3. Varieties of (non-associative) algebras

Linearization of identities

[Zherlakov, Shestakov, Slin'ko, Shirshov: Rings that are nearly associative]

The Book of the Four



$\Bbbk$ fixed field

$n=1$

Algebra: $(A, \mu_1, \dots, \mu_n)$

A: vector space over $\Bbbk$

$\mu_1, \dots, \mu_n$: multi-linear operations — language (signature)

We will mainly deal with **binary** case:

$$\mu_i : A \otimes A \longrightarrow A$$

$$a \otimes b \longmapsto a \cdot_i b$$

For a fixed $\Bbbk$ and $B = (\mu_1, \dots, \mu_n)$

(symbols of operations)

$\downarrow$

class of all B-algebras over $\Bbbk$

Subalgebras

Homomorphisms $\varphi : A_1 \to A_2$ $\quad \mathrm{Hom}_{alg}(A_1, A_2)$

Direct (Cartesian) products $\prod_i A_i$

Ideals $I \lhd A$

Quotients A/I $\quad \tau : A \to A/I$ hom

...

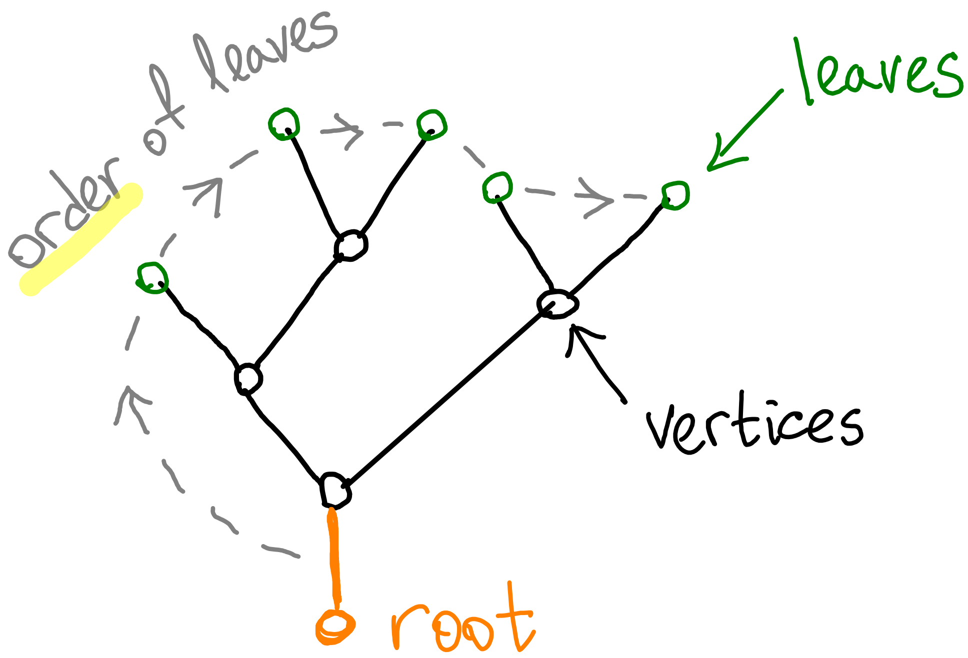
...

$\mathbb{T}$ = set of planar rooted binary trees (up to isotopy)

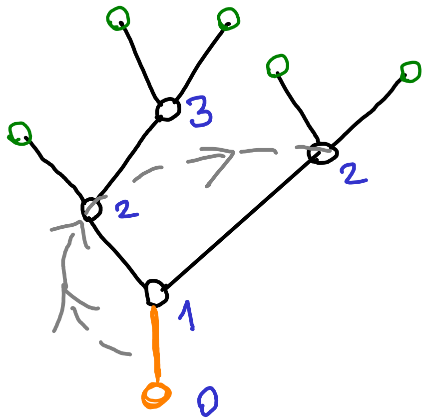
n vertices

$n+1$ leaves

root



height of a vertex:



Planar $\Rightarrow$ vertices of the same height are ordered

Fix $B = (\mu_1, \dots, \mu_n)$

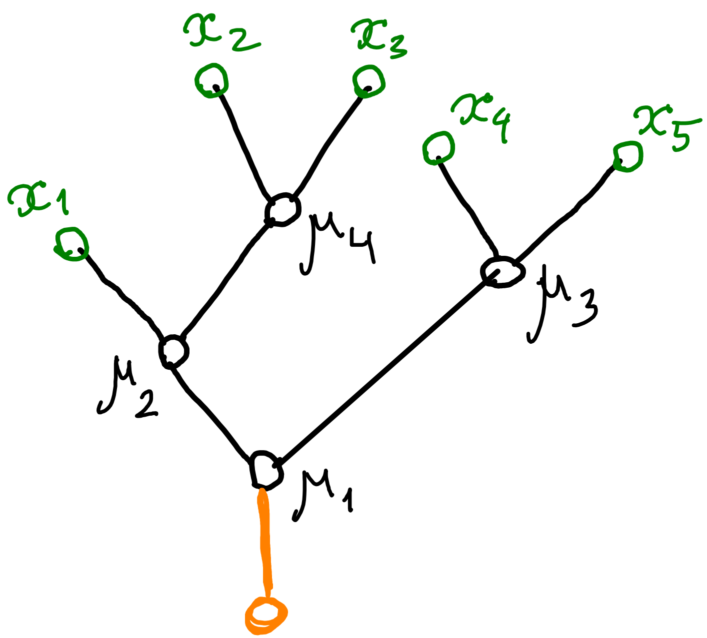
$X \neq \emptyset$ set

$\mathbb{T}_B(X) :=$ set of all planar rooted binary trees with labels:

vertices $\leftarrow B$

leaves $\leftarrow X$

root has no label

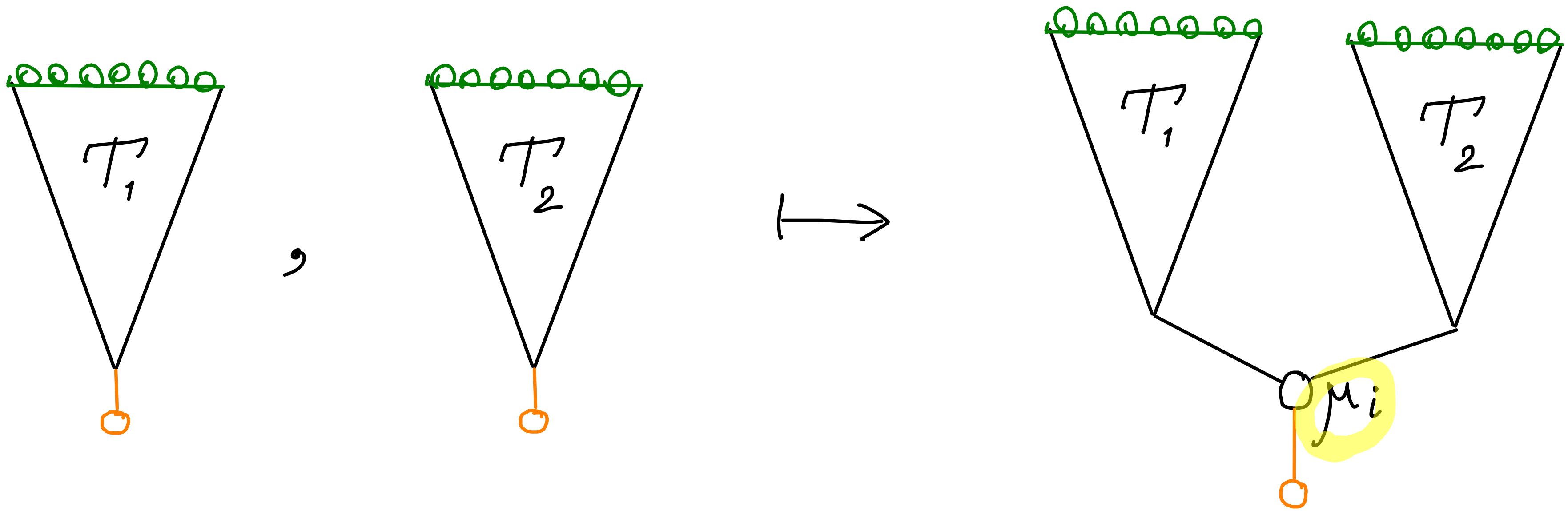


Later on,

$X = \{x_1, x_2, x_3, \dots\}$

Define binary operations μ_i on $\mathcal{T}_B(X)$:

$\mu_i \in B$



Extend by linearity to

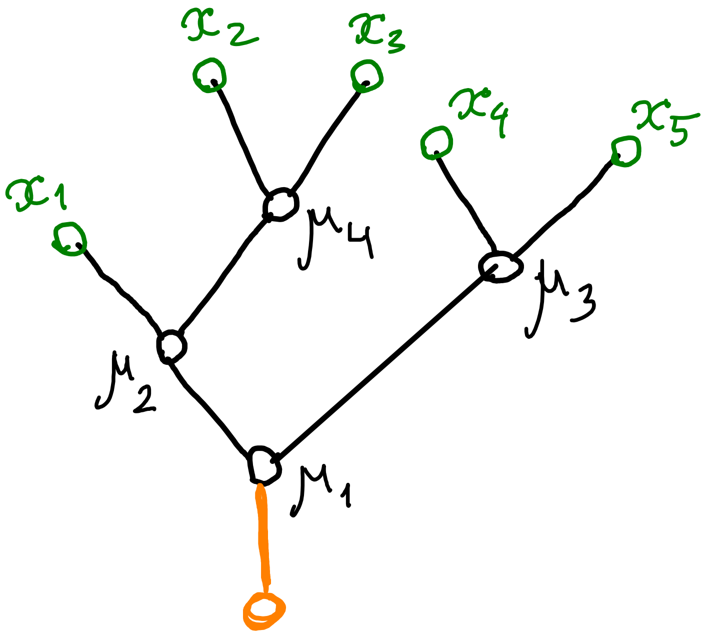
$$\mathcal{F}_B(X) = \mathrm{span}_k\, \mathcal{T}_B(X)$$

formal linear span of the set of trees

$\Downarrow$

$\mathcal{F}_B(X)$ free algebra of the language $(\mu_i)_{i=1}^n$

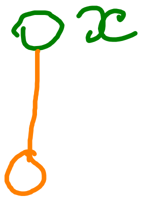
tree generated by X



$\longleftrightarrow \quad (x_1 \cdot_2 (x_2 \cdot_4 x_3)) \cdot_1 (x_4 \cdot_3 x_5)$

non-associative word

$\mathcal{F}_B(X) \longleftrightarrow$ non-associative polynomials in X with operations μ_i

In particular,  $\longleftrightarrow x$ so $X \hookrightarrow F_B(X)$

$$
\begin{tikzcd}
X \arrow[rr, "\subseteq"] \arrow[drr, "\forall \text{ map}"'] & & F_B(X) \arrow[d, dashed, "\exists! \text{ homomorphism}"] \\
 & & A
\end{tikzcd}
$$

$x_i \mapsto a_i \in A$

Universal property of the free algebra

$$(x_1 \cdot_2 (x_2 \cdot_4 x_3)) \cdot_1 (x_4 \cdot_3 x_5) \longmapsto (a_1 \cdot_2 (a_2 \cdot_4 a_3)) \cdot_1 (a_4 \cdot_3 a_5) \in A$$

$f \in \mathcal{F}_B(X)$ is an identity on A

$A \models f$

if

$$\begin{tikzcd} X \arrow[r, "\subseteq"] \arrow[dr, "\forall \text{ map}"'] & \mathcal{F}_B(X) \arrow[d, dashed] \\ & A \end{tikzcd} \qquad f \mapsto 0$$

$$Id(A) = \{ f \mid A \models f \} = \bigcap \text{kernel} \lhd \mathcal{F}_B(X)$$

all identities of A

not just an ideal:

$f \in Id(A) \Rightarrow \varphi(f) \in Id(A)$

for every $\varphi \in Hom_{alg.}\big(\mathcal{F}_B(A), \mathcal{F}_B(A)\big)$

$Id(A)$ is a T-ideal

transformation

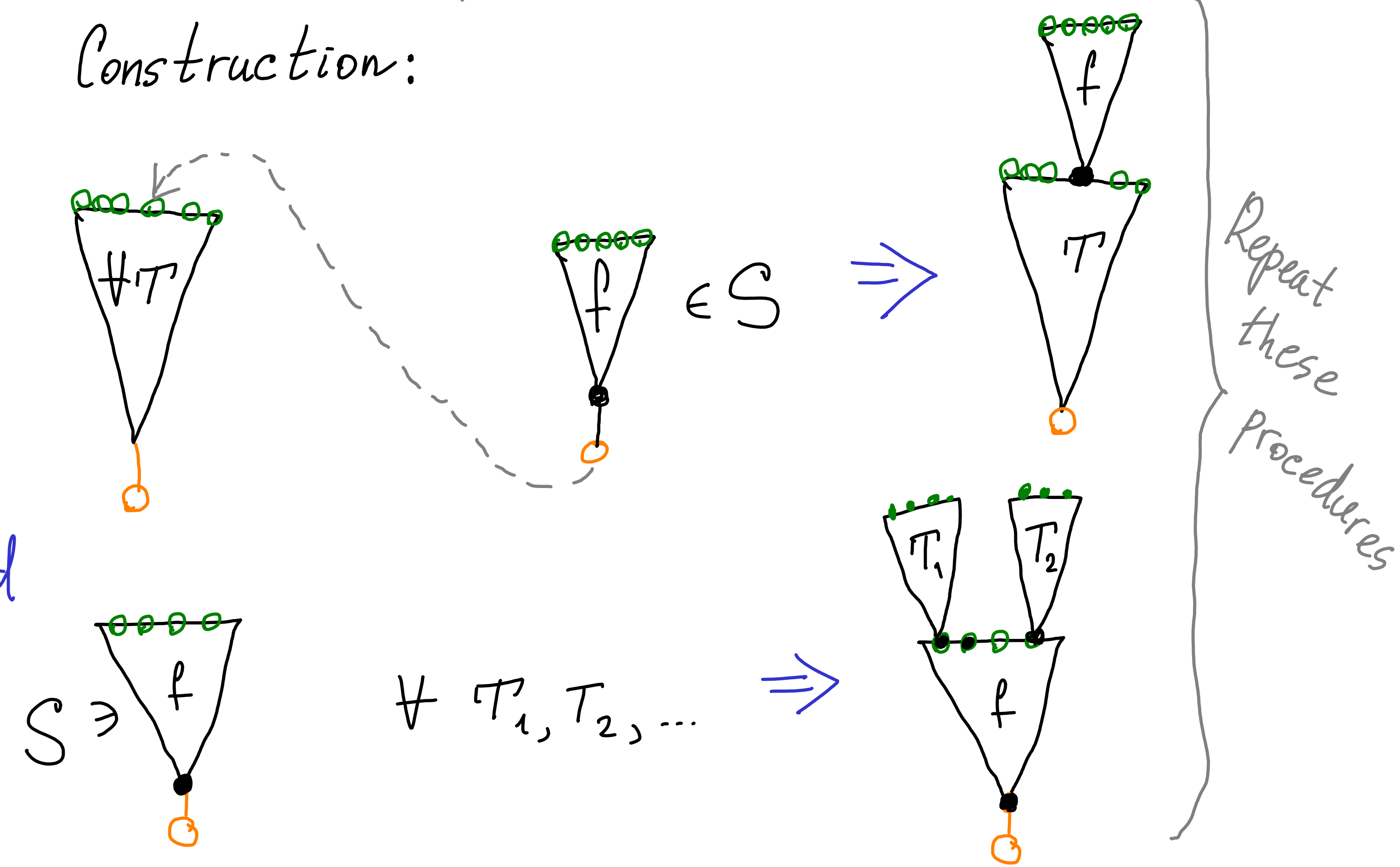
Given a subset $S \subseteq \mathcal{F}_B(X)$

T-ideal(S) is the smallest T-ideal $\supseteq S$

(generated by S)

If $|S|<\infty$ then the T-ideal is finitely based

Construction:



and

Example $A = M_2(\mathbb{C})$ associative algebra language (μ_1)

$$as = (x_1x_2)x_3 - x_1(x_2x_3) \in Id(A)$$

But $T\text{-ideal}(as) \subsetneq Id(A)$

generated by as

$$st_4 = \sum_{\sigma \in S_4} (-1)^\sigma x_{1\sigma} x_{2\sigma} x_{3\sigma} x_{4\sigma} \in Id(A) \setminus T\text{-ideal}(as)$$

in $x_1, \dots, x_4$

(!) multi-linear

$\Downarrow$

Enough to check for

$e_{ij} e_{kl} = \delta_{jk} e_{il}$

$$x_1 = e_{i_1 j_1}, \dots, x_4 = e_{i_4 j_4}$$

- Over an infinite field $\Bbbk$, every T-ideal is generated by homogeneous (non-associative) polynomials

  same deg w.r.t each x_i

If $A \models f_0 + f_1 + \dots + f_m$, $\deg_x f_i = i$, then $f_i \dashv A$

Follows from van-der-Monde's argument

- Doesn't work over finite fields: $A = GF(p^n)$ over $\Bbbk = \mathbb{Z}_p$

  $f = x^{p^n} - x \dashv A$ non-homogeneous

- Over a field char $= 0$, every T-ideal is generated by multi-linear polynomials

homogeneous of deg $= 1$ w.r.t each x_i

Linearization procedure

$$f \in \mathcal{F}_B(x_1, x_2, \dots) \longmapsto Lf(x_1, \dots, x_n) \quad \text{or } \Delta f$$

homogeneous of deg $= n$

multi-linear: linearization of f

so that $A \models f \iff Lf \dashv A$

$\Longrightarrow$ always

$\Longleftarrow$ if char $\Bbbk = 0$ (or $> n$)

Example

char $\Bbbk = 0$ (or > 2)

language: $B = (\mu_1)$ $\qquad x \cdot_1 y = xy$

$$f = x_1^2 \in \mathcal{F}_B(x_1, x_2, \dots)$$

if $2 \neq 0$ in $\Bbbk$

Suppose $A \vDash f$: $a^2 = 0 \quad \forall a \in A$

Then $\forall\ a, b \in A$ $\quad (a+b)^2 = 0$

$$\left.\begin{array}{l} (a+b)^2 = 0 \\ \quad \| \\ \underbrace{a^2}_{=0} + ab + ba + \underbrace{b^2}_{=0} \end{array}\right\} \Rightarrow x_1x_2 + x_2x_1 \vDash A$$

$$\boxed{x_1x_1 \longmapsto x_1x_2 + x_2x_1}$$

Example

char $\Bbbk = 0$ (or > 2)

language: $B = (\mu_1)$ $x \cdot_1 y = xy$

$$f = \text{l-alt} = (x_1 x_1) x_2 - x_1 (x_1 x_2) \in \mathcal{F}_B(x_1, x_2, \dots)$$

(left) alternative

$x_1 = a + b$, $x_2 = c$

$$\Downarrow$$

$$Lf = \underbrace{(x_1 x_2) x_3}_{} + \underbrace{(x_2 x_1) x_3}_{} - \underbrace{x_1 (x_2 x_3)}_{} - \underbrace{x_2 (x_1 x_3)}_{}$$

$$= \underbrace{(x_1, x_2, x_3)}_{} + \underbrace{(x_2, x_1, x_3)}_{}$$

where $(a, b, c) = (ab)c - a(bc)$ is the associator

$$\boxed{(x_1, x_1, x_2) \longmapsto (x_1, x_2, x_3) + (x_2, x_1, x_3)}$$

Exercise 1) Find the linearization of right alternative identity

$$\text{r-alt} = (x_1 x_2) x_2 - x_1 (x_2 x_2)$$

Alternative algebra: $A \models \text{l-alt}, \text{r-alt}$

Exercise 2) Prove

$$(x_1, x_2, x_3) - (x_2, x_3, x_1) \in T\text{-ideal}(\text{l-alt}, \text{r-alt})$$

$$(x_1 x_2) x_1 - x_1 (x_2 x_1) \in T\text{-ideal}(\text{l-alt}, \text{r-alt})$$

Hint: linearize first ...

Example $\quad f=(x_1x_1)x_1 \qquad \deg = 3$

If $A\models f$, then $\quad x_1=a+b$

$$0=f(a+b)=(aa+ab+ba+bb)(a+b)$$

$$\begin{aligned}&=(aa)a+(ab)a+(ba)a+(bb)a\\&+(aa)b+(ab)b+(ba)b+(bb)b \qquad \forall a,b\in A\end{aligned}$$

$$\begin{aligned}&=(ab)a+(ba)a+(aa)b\\&+(bb)a+(ab)b+(ba)b\end{aligned}$$

$$A\models \underbrace{(x_1x_2)x_1+(x_2x_1)x_1+(x_1x_1)x_2}+\underbrace{(x_2x_2)x_1+(x_1x_2)x_2+(x_2x_1)x_2}$$

homogeneous components by $\deg_{x_1}$

Hence

$$(x_1x_1)x_1 \Dashv A \;\Longrightarrow\; A \models \overbrace{(x_1x_2)x_1 + (x_2x_1)x_1 + (x_1x_1)x_2}^{\text{partial linearization}}$$

$$\Longleftarrow \quad \text{if } \operatorname{char} \mathbb{k} \neq 3$$

linearize again

$$Lf = (x_1x_2)x_3 + (x_3x_2)x_1 + (x_2x_1)x_3 + (x_2x_3)x_1 + (x_1x_3)x_2 + (x_3x_1)x_2$$

$$= \sum_{\sigma \in S_3} (x_{1\sigma}\, x_{2\sigma})\, x_{3\sigma}$$

$$x_1^3 \Dashv A \;\Longrightarrow\; Lf \Dashv A$$

$$\Longleftarrow \quad \text{if } \operatorname{char} \neq 2, 3$$

Exercise

Find Lf for

$$f = (x_1x_1)(x_1x_2) - x_1((x_1x_1)x_2)$$

Jordan identity

$\neq$ Jordan matrix ...

Variety is a subclass of B-algebras over k closed under

- homomorphic images H
- subalgebras S
- Cartesian products P

Birkhoff's Theorem

Var is a variety $\Longrightarrow$ exists a T-ideal $I_{Var} \triangleleft F_B(x_1, x_2, \dots)$ such that

$$A \in Var \iff A \models f \ \forall f \in I_{Var}$$

$$\boxed{I_{Var} = \bigcap_{A \in Var} Id(A)}$$

$\Longleftarrow$ Exercise

- Over a field char $= 0$, I_{Var} is generated by multi-linear polynomials (defining identities of Var)

Examples

$B = (\mu_1)$ one binary operation

Var = As : $I_{As} = T\text{-ideal}\big(as = (x_1x_2)x_3 - x_1(x_2x_3)\big)$

Com : $I_{Com} = T\text{-ideal}\big(x_1x_2 - x_2x_1,\ as\big)$

Leib : $I_{Leib} = T\text{-ideal}\big(jac = as + x_2(x_1x_3)\big)$

Lie : $I_{Lie} = T\text{-ideal}\big(x_1x_2 + x_2x_1,\ jac\big)$

Examples

$B = (\mu_1)$ one binary operation

Var = Alt : $I_{Alt} = T\text{-ideal}(\ell\text{-alt},\ r\text{-alt})$

LSym : $I_{LSym} = T\text{-ideal}(\ell\text{-sym})$

aka preLie

$$\ell\text{-sym} = as(x_1,x_2,x_3) - as(x_2,x_1,x_3)$$

Similarly : RSym

Zinb : $I_{Zinb} = T\text{-ideal}\left(as + (x_2x_1)x_3\right)$

Zinbiel algebras c.f. Leibniz:

aka preCom

$$jac = as + x_2(x_1x_3)$$

Explain later via operads

$\mathbb{k}$ field

$B = (\mu_1, \dots, \mu_m)$ binary language

Var variety of B-algebras

Var is defined by multi-linear identities

$$\mathrm{Var}\langle x_1, x_2, \dots \rangle = \mathcal{F}_B(x_1, x_2, \dots) / I_{\mathrm{Var}}$$

(relatively) free Var-algebra generated by $X = \{x_1, x_2, \dots\}$

Universal property in Var

$$\begin{tikzcd} X \arrow[r, "\subseteq"] \arrow[dr, "\forall"'] & \mathrm{Var}\langle X \rangle \arrow[d, dashed, "\exists!"] \\ & A \end{tikzcd}$$

$A \in \mathrm{Var}$

Multi-linear component

$X = \{x_1, x_2, \dots\}$

$$\mathrm{Var}(n) = \left\{ \begin{array}{c} \text{images of all multi-linear} \\ f(x_1, \dots, x_n) \in \mathcal{F}_{\mathcal{B}}(X) \end{array} \right\} \subseteq \mathrm{Var}\langle X \rangle$$

$n = 1, 2, 3, \dots$

Examples

$$As(n) = \mathrm{span}\{ x_{1\sigma} \dots x_{n\sigma} \mid \sigma \in S_n \} \qquad \dim As(n) = n!$$

$$Com(n) = \mathrm{span}\{ x_1 \dots x_n \} \qquad \dim Com(n) = 1$$

$$Leib(n) = \mathrm{span}\{ x_{1\sigma}(x_{2\sigma} \dots (x_{(n-1)\sigma} x_{n\sigma}) \dots) \mid \sigma \in S_n \} \qquad \dim Leib(n) = n!$$

Examples

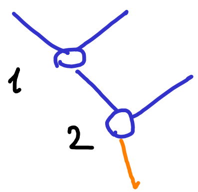
$$\mathrm{Lie}(n) = \mathrm{span}\left\{ x_{1\sigma}(x_{2\sigma} \cdots (x_{(n-1)\sigma}\, x_n)\cdots) \;\middle|\; \sigma \in S_{n-1} \right\}$$

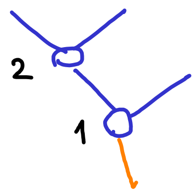
$$\dim \mathrm{Lie}(n) = (n-1)!$$

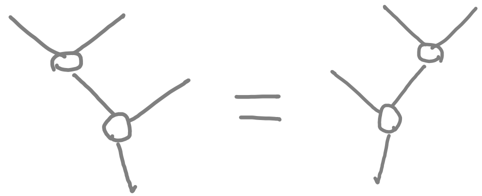
$$\mathrm{LSym}(n) = \mathrm{span}\left\{ \begin{array}{l} \text{rooted trees (not planar)} \\ \text{with } n \text{ enumerated vertices} \end{array} \right\}$$

pre Lie

$n=2$:

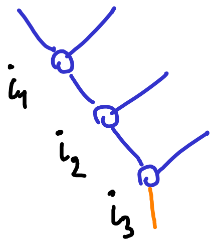




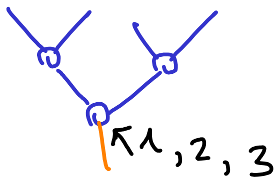


$$\dim \mathrm{LSym}(n) = n^{n-1}$$

$n=3$:



×6



×3

[Chapoton, Livernet, 2001]

goes back to Cayley, 1890

Examples

$$\mathrm{Zinb}(n) = \mathrm{span}\left\{ (\dots(x_{1\sigma}x_{2\sigma})\dots x_{n\sigma}) \;\middle|\; \sigma \in S_n \right\}$$

$$\dim \mathrm{Zinb}(n) = n!$$

$$x(yz) \to (xy)z + (yx)z$$

$$\mathrm{Alt}(n) = ?$$

Find $\mathbf{Alt(3)} = \left\{ \text{All monomials of } \deg = 3 \right\} \Big/ \begin{pmatrix} l\text{-alt} \\ r\text{-alt} \end{pmatrix}$

$\dim = 12$

$$\left.\begin{aligned} l\text{-alt} &= (x_1, x_2, x_3) + (x_2, x_1, x_3) \\ r\text{-alt} &= (x_1, x_2, x_3) + (x_1, x_3, x_2) \end{aligned}\right\} \text{ \& all permutations of } x_1, \dots, x_3$$

$$(a,b,c) = (ab)c - a(bc)$$

Denote:

$$\underset{a_1}{(x_1x_2)x_3} \qquad \underset{a_2}{(x_2x_1)x_3} \qquad \underset{a_3}{x_3(x_1x_2)} \qquad \underset{a_4}{x_3(x_2x_1)}$$

$$\underset{b_1}{(x_3x_2)x_1} \qquad \underset{b_2}{(x_2x_3)x_1} \qquad \underset{b_3}{x_1(x_3x_2)} \qquad \underset{b_4}{x_1(x_2x_3)} \qquad b_i = a_i^{(13)}$$

$$\underset{c_1}{(x_1x_3)x_2} \qquad \underset{c_2}{(x_3x_1)x_2} \qquad \underset{c_3}{x_2(x_1x_3)} \qquad \underset{c_4}{x_2(x_3x_1)} \qquad c_i = a_i^{(23)}$$

$$\text{l-alt} = (x_1, x_2, x_3) + (x_2, x_1, x_3)$$

$$= (x_1x_2)x_3 - x_1(x_2x_3) + (x_2x_1)x_3 - x_2(x_1x_3) = a_1 - b_4 + a_2 - c_3$$

$$\text{r-alt} = (x_1, x_2, x_3) + (x_1, x_3, x_2)$$

$$= (x_1x_2)x_3 - x_1(x_2x_3) + (x_1x_3)x_2 - x_1(x_3x_2) = a_1 - b_4 + c_1 - b_3$$

We have to add transformations:

$$\ell\text{-alt}^{(13)} \quad \ell\text{-alt}^{(23)}$$

$$r\text{-alt}^{(12)} \quad r\text{-alt}^{(13)}$$

since $\ell\text{-alt}^{(12)} = \ell\text{-alt}$

$r\text{-alt}^{(23)} = r\text{-alt}$

S_3: Stabilizer of order 2

$\Downarrow$

$|\text{Orbit}| = \frac{3!}{2} = 3$

- $\ell\text{-alt}^{(13)} = \left(a_1 - b_4 + a_2 - c_3\right)^{(13)} = b_1 - a_4 + b_2 - c_4$
- $\ell\text{-alt}^{(23)} = \left(a_1 - b_4 + a_2 - c_3\right)^{(23)} = c_1 - b_3 + c_2 - a_3$
- $r\text{-alt}^{(12)} = \left(a_1 - b_4 + c_1 - b_3\right)^{(12)} = a_2 - c_3 + b_2 - c_4$
- $r\text{-alt}^{(13)} = \left(a_1 - b_4 + c_1 - b_3\right)^{(13)} = b_1 - a_4 + c_2 - a_3$

span{ • • • • • • } = ?

```
int n=4;
ring r=0,(a(1..n),b(1..n),c(1..n)),lp;

ideal TAlt =
        a(1)-b(4)+a(2)-c(3),
        a(1)-b(4)+c(1)-b(3),
        b(1)-a(4)+b(2)-c(4),
        c(1)-b(3)+c(2)-a(3),
        a(2)-c(3)+b(2)-c(4),
        b(1)-a(4)+c(2)-a(3);
```

```
> std(TAlt);

_[1]=b(2)-b(3)+c(1)-c(4)

_[2]=a(4)-b(1)-b(2)+c(4)

_[3]=a(3)+a(4)-b(1)-c(2)

_[4]=a(2)+b(2)-c(3)-c(4)

_[5]=a(1)-b(3)-b(4)+c(1)
```

$$\boxed{\begin{gathered} \mathrm{Alt}(3) = \mathrm{span}\{c_1, c_2, c_3, c_4, b_1, b_3, b_4\} \\ \dim \mathrm{Alt}(3) = 7 \end{gathered}}$$

[Dzhumadil'daev, Zusmanovich, 2013]:

n	1	2	3	4	5	6
$\mathrm{Alt}(n)$	1	2	7	32	175	1080

Example $B = \{\mu_1, \mu_2\}$ $\qquad \mu_1(a,b) = ab$

$\mu_2(a,b) = \{a,b\}$

$Var = \mathbf{Pois}$:

$$I_{Pois} = T\text{-ideal}\Big(x_1x_2 - x_2x_1, \ \ \mathbf{as}^{\cdot},$$

$$\{x_1, x_2\} + \{x_2, x_1\}, \ \ \mathbf{jac}^{\{\,\}},$$

$$\mathbf{leib} = \{x_1, x_2x_3\} - \{x_1x_2\}x_3 - x_2\{x_1, x_3\}\Big)$$

$\dim \mathrm{Pois}(n) = n!$ $\qquad \Leftarrow \mathrm{PBW}$