Stable Theories with a Finite Number of Countable Models

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Let T be a countable complete theory. As usual by $I(T,\omega)$ we denote the number of pairwise non-isomorphic countable models. Obviously, if $I(T,\omega)<\omega$ then T has few types ($|S(T)|=\omega$) and for any type $p\in S(T)$ there exists a prime model $\mathcal{M}_p$ over a realization of p.

For any types $p,q\in S(T)$ we shall write $p\leqslant_{RK} q$ and say that p *is not more than* q *relative to the Rudin—Keisler preorder* (see [**1**]) or p *is subordinated to* q if $\mathcal{M}_q$ has a realization of type p. At the same time we shall write $\mathcal{M}_p\leqslant_{RK}\mathcal{M}_q$ and say that $\mathcal{M}_p$ *is subordinated to* $\mathcal{M}_q$ if $p\leqslant_{RK} q$. By $\mathrm{RK}(T)$ we denote the set of all isomorphism types of models $\mathcal{M}_p$ with the relation of subordination induced by the relation $\leqslant_{RK}$ for models $\mathcal{M}_p$.

We say that models $\mathcal{M}_p$ and $\mathcal{M}_q$ are *reciprocally subordinated* if $\mathcal{M}_p\leqslant_{RK}\mathcal{M}_q$ and $\mathcal{M}_q\leqslant_{RK}\mathcal{M}_p$. Isomorphism types $\mathbf{M}_1$ and $\mathbf{M}_2$ from $\mathrm{RK}(T)$ are *reciprocally subordinated* ($\mathbf{M}_1\sim_{RK}\mathbf{M}_2$) if their representatives are reciprocally subordinated.

A model $\mathcal{M}$ is said to be *(strongly) limit over a type* p if $\mathcal{M}$ is a union of elementary chain $(\mathcal{M}_n)_{n\in\omega}$ such that $\mathcal{M}_n\simeq\mathcal{M}_p$, $n\in\omega$, and $\mathcal{M}\not\simeq\mathcal{M}_p$.

Let $\mathrm{RK}(T)$ be a finite system. For any class $\widetilde{\mathbf{M}}\in\mathrm{RK}(T)/{\sim_{RK}}$ consisting of isomorphism types of reciprocally subordinated models $\mathcal{M}_{p_1},\ldots,\mathcal{M}_{p_n}$ we denote by $\mathrm{IL}(\widetilde{\mathbf{M}})$ the number of pairwise non-isomorphic limit models each of which is limit over some type p_i.

Remember [**2**] that a type $p\in S(T)$ is said to be *powerful* if $\mathcal{M}_p$ has a realization of any type from $S(T)$.

The following theorem can be considered as a syntactical characterization of the class of countable complete theories having a finite number of countable models.

Theorem 1. *For any countable copmlete theory T the following conditions are equivalent*:

(1) $I(T,\omega)<\omega$,

(2) *T has few types,* $|\mathrm{RK}(T)|<\omega$ *and* $\mathrm{IL}(\widetilde{\mathbf{M}})<\omega$ *for any* $\widetilde{\mathbf{M}}\in\mathrm{RK}(T)/{\sim_{RK}}$.

If the condition (1) *(or* (2)*) is true, then T satisfies the following*:

(a) $\mathrm{RK}(T)$ *has the least element* $\mathbf{M}_0$ (*the isomorphism type of prime model*) *and* $\mathrm{IL}(\widetilde{\mathbf{M}_0}) = 0$,

(b) $\mathrm{RK}(T)$ *has the greatest* $\sim_{RK}$-*class* $\widetilde{\mathbf{M}_1}$ (*the class of isomorphism types of all prime models over realizations of powerful types*), *and* $|\,\mathrm{RK}(T)| > 1$ *implies* $\mathrm{IL}(\widetilde{\mathbf{M}_1}) \geqslant 1$,

(c) *if* $|\widetilde{\mathbf{M}}| > 1$ *then* $\mathrm{IL}(\widetilde{\mathbf{M}}) \geqslant 1$.

Moreover the following decomposition formula is true:

$$I(T,\omega) = |\,\mathrm{RK}(T)| + \sum_{i=0}^{|\mathrm{RK}(T)/\sim_{RK}|-1} \mathrm{IL}(\widetilde{\mathbf{M}_i}),$$

where $\widetilde{\mathbf{M}_0}, \ldots, \widetilde{\mathbf{M}_{|\mathrm{RK}(T)/\sim_{RK}|-1}}$ *are all elements of partially ordered set* $\mathrm{RK}(T)/{\sim_{RK}}$.

Using a modification of Hrushovski's style construction [**3**] of generic trigonometry having small stable theory and an infinite weight the author proved that any described possibility of preorder $\mathrm{RK}(T)$ and of *distribution function* of numbers of limit models for $\sim_{RK}$-classes can be realized in the class of stable theories.

Theorem 2 (Classification Theorem). *Let* $\langle X, \leqslant\rangle$ *be a finite preordered set with the least element* x_0 *and the greatest class* $\widetilde{x_1}$ *in the ordered factor-set* $\langle X, \leqslant\rangle/{\sim}$ *by the relation* $\sim$ (*where* $x \sim y \Leftrightarrow x \leqslant y$ *and* $y \leqslant x$), $f : X/{\sim} \to \omega$ *be a function satisfying the following conditions*:

(a) $f(\widetilde{x_0}) = 0$,

(b) $|X| > 1$ *implies* $f(\widetilde{x_1}) \geqslant 1$,

(c) $|\widetilde{y}| > 1$ *implies* $f(\widetilde{y}) \geqslant 1$.

Then there exists a stable theory T *and an isomorphism* $g : \langle X, \leqslant\rangle \tilde{\to} \mathrm{RK}(T)$ *such that* $\mathrm{IL}(g(\widetilde{y})) = f(\widetilde{y})$ *for any* $\widetilde{y} \in X/{\sim}$.

From the Classification Theorem we get the solution of known problem of the existence of stable theory having a finite number of countable models.

Corollary 3. *For any natural number* $n \geqslant 3$ *there exists a stable theory* T *such that* $I(T,\omega) = n$.

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References

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