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On derivations of the ternary Malcev algebra M_8

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Ternary Malcev algebras are a particular case of n-ary Malcev algebras, first defined in [3], and these naturally arise from the classification of n-ary vector cross product algebras [1]. Indeed, the classification theorem for the latter asserts that, in the case $n = 2$, the only possible algebras are the simple 3-dimensional Lie algebra $sl(2)$ and the simple 7-dimensional Malcev algebra C_7; in the case $n \geq 3$, those are the simple $(n+1)$-dimensional n-Lie algebras (which, in turn, are a natural generalization of Lie algebras to the case of an n-ary multiplication [2], and nowadays called *Filippov algebras*) with vector cross product, being analogues of $sl(2)$, and also some exclusive ternary algebras arising on composition algebras.

It has been proved [3] that the latter are ternary central simple Malcev algebras, which are not 3-Lie algebras if the characteristic of the ground field is different from 2 and 3 (more generally, the result states that every n-ary vector cross product algebra is an n-ary central simple Malcev algebra).

The class of n-ary Malcev algebras has also the following interesting properties:
1. It is an extension of the class of n-Lie algebras, *i.e.*, every n-Lie algebra is an n-ary Malcev algebra (generalizing the fact that every Lie algebra is a Malcev algebra);
2. Fixing a component in the multiplication (*i.e,* defining a new reduced operation on the vector space A of the n-ary Malcev algebra by the rule $[x_1, \dots, x_{n-1}]_a = [a, x_1, \dots, x_{n-1}]$), we obtain an $(n-1)$-ary Malcev algebra.

By an n-ary Jacobian, we mean the following function defined on an

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n-ary algebra:

$$J(x_1,\dots,x_n;y_2,\dots,y_n) =$$
$$[[x_1,\dots,x_n],y_2,\dots,y_n] - \sum_{i=1}^{n}[x_1,\dots,[x_i,y_2,\dots,y_n],\dots,x_n].$$

An n-ary *Malcev algebra* ($n \geq 3$) is an Ω-algebra L with one anticommutative n-ary operation $[x_1,\dots,x_n]$ satisfying the identity

$$-J(zR_x,x_2,\dots,x_n;y_2,\dots,y_n) = J(z,x_2,\dots,x_n;y_2,\dots,y_n)R_x,$$

where $R_x = R_{x_2,\dots,x_n}$ is the operator of right multiplication.

Henceforth, we assume that Φ is a field of characteristic not equal to $2,3$ and denote by A a composition algebra over Φ with an involution $^{-}:$ $a \mapsto \bar{a}$ and nonsingular symmetric bilinear form $\langle x,y\rangle = \frac{1}{2}(x\bar{y}+y\bar{x})$. If A is equipped with a ternary multiplication $[\cdot,\cdot,\cdot]$ by the rule

$$[x,y,z] = x\bar{y}z - \langle y,z\rangle\, x + \langle x,z\rangle\, y - \langle x,y\rangle\, z,$$

then A becomes a ternary Malcev algebra which will be denoted by $M(A)$. If $dim\, A = 8$ then $M(A)$ is not a 3-Lie algebra and we denote it by M_8.

Let $\mathcal{R}$ be the vector space generated by the right multiplications of M_8. Let $Ass(\mathcal{R})$ and $Lie(\mathcal{R})$ denote, respectively, the associative and the Lie algebra generated by $\mathcal{R}$. Let $Der(M_8)$ be the derivation algebra of M_8. Recall that a derivation is called *inner* if it belongs to the Lie algebra $Lie(\mathcal{R})$ of transformations.

Theorem 1. *1.* $Ass(\mathcal{R}) = M_{8,8}(\Phi) = \langle\mathcal{R}^2\rangle$*;*

2. $Lie(\mathcal{R}) \cong D_4$ *and* $Lie(\mathcal{R}) = \mathcal{R}$ *as vector spaces;*

3. $Der(M_8) \cong B_3$.

Theorem 2. *All derivations of* M_8 *are inner.*

In the case of Malcev algebras we know that the operators of the type $[R_x,R_y]+R_{xy}$ are inner derivations.

Theorem 3. *Let* $M(A)$ *be a ternary Malcev algebra. For any* $x,y,z \in A$

$$[R_{z,x},R_{z,y}] + R_{z,[x,z,y]} \in Der(M(A)).$$

Let A be an n-ary anticommutative algebra with multiplication $[\cdot,...,\cdot]$. Every operator $D: A \longrightarrow A$ such that

$$[D,R_a] \in Lie\,(\mathcal{R})\,,\ \text{for all } R_a \in Lie\,(\mathcal{R})\,,$$

is said to be a *quasi-derivation* of A. The set of all quasi-derivations of the algebra A we denote by $QDer(A)$.

Theorem 4. $QDer(M_8) = \langle Id\rangle_{\Phi} \oplus Lie(\mathcal{R})$.

In the proofs of all these results we use some symmetries of the canonical basis of a composition algebra.

References

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