Mal′tsev Meeting
Novosibirsk (2006)
Abstracts

On the finite closure property for fusions of generic classes

S. V. Sudoplatov

E. Hrushovski [**1**] has defined a mechanism of fusion of two generic theories to get a strongly minimal theory having a structure with fields of two different characteristics. His technics has received recently an essential development in the connection with questions of existence of fusions of generic structures having various given properties [**2**]–[**7**]. Considered in [**8**] combinations and colorings of countable homogeneous models as well as Ehrenfeucht generic structures [**9**] can be interpreted as special cases of fusions.

As the saturation of generic models is caused by formula definability of operation of self-sufficient closures $\overline{A}$ for each finite set A, there is a natural question about possibility of construction of fusions of generic theories, having formula definable operations of self-sufficient closures, with the condition of formula definability of resulting operation of self-sufficient closures. We shall offer a sufficient condition of existence of such fusions.

Further we shall use the terminology from [**10**].

Let $(\mathbf{T};\leqslant)$ be a generic class. We say that $(\mathbf{T};\leqslant)$ has *the finite closure property* if any model of $(\mathbf{T};\leqslant)$-generic theory has finite closures.

Let $(\mathbf{T}_0;\leqslant_0)$, $(\mathbf{T}_1;\leqslant_1)$ and $(\mathbf{T}_2;\leqslant_2)$ be generic classes of languages Σ_0, Σ_1 and Σ_2 accordingly, $\Sigma_0=\Sigma_1\cap\Sigma_2$, $\leqslant_0\,=\,\leqslant_1\cap\leqslant_2$. A *fusion* of the classes $(\mathbf{T}_1;\leqslant_1)$ and $(\mathbf{T}_2;\leqslant_2)$ over the class $(\mathbf{T}_0;\leqslant_0)$ is a generic class $(\mathbf{T}_3;\leqslant_3)$ of language $\Sigma_1\cup\Sigma_2$ such that $(\mathbf{T}_3;\leqslant_3)\restriction\Sigma_i=(\mathbf{T}_i;\leqslant_i)$, $i=1,2$. Thus a $(\mathbf{T}_3;\leqslant_3)$-generic model (theory) is called a *fusion* of $(\mathbf{T}_1;\leqslant_1)$-generic and $(\mathbf{T}_2;\leqslant_2)$-generic models (theories).

We denote fusions of $(\mathbf{T}_1;\leqslant_1)$ and $(\mathbf{T}_2;\leqslant_2)$ over $(\mathbf{T}_0;\leqslant_0)$ by

$$(\mathbf{T}_1;\leqslant_1)\,\mathcal{F}_{(\mathbf{T}_0;\leqslant_0)}\,(\mathbf{T}_2;\leqslant_2).$$

A fusion of $(\mathbf{T}_1;\leqslant_1)$-generic model $\mathcal{M}_1$ (theory T_1) and of $(\mathbf{T}_2;\leqslant_2)$-generic model $\mathcal{M}_2$ (theory T_2) over a $(\mathbf{T}_0;\leqslant_0)$-generic model $\mathcal{M}_0$ (theory T_0) is denoted by $\mathcal{M}_1\,\mathcal{F}_{\mathcal{M}_0}\,\mathcal{M}_2$ ($T_1\,\mathcal{F}_{T_0}\,T_2$). We may assume that $\mathcal{M}_0$ is an elementary submodel of $\mathcal{M}_1\restriction\Sigma_0$ and of $\mathcal{M}_2\restriction\Sigma_0$, and $\mathcal{M}_1$, $\mathcal{M}_2$ are elementary submodels of $\mathcal{M}_3\restriction\Sigma_1$ and $\mathcal{M}_3\restriction\Sigma_2$ accordingly. Thus, $\mathcal{M}_1\preccurlyeq(\mathcal{M}_1\mathcal{F}_{\mathcal{M}_0}\mathcal{M}_2)\restriction\Sigma_1$ and $\mathcal{M}_2\preccurlyeq(\mathcal{M}_1\mathcal{F}_{\mathcal{M}_0}\mathcal{M}_2)\restriction\Sigma_2$.

Denote by Cl_i operations of self-sufficient closures in the models $\mathcal{M}_i$, $i=1,2,3$. It is obvious, that for any finite set $A\subseteq M_3$ we have $\mathrm{Cl}_3(A)\supseteq\bigcup\limits_{n\in\omega}A_n$, where

The author was supported by the Russian Foundation for Basic Researches (Grant 05-01-00411).

$A_0 = A$, $A_{n+1} = \mathrm{Cl}_1(\mathrm{Cl}_2(A_n))$. Moreover, if $\mathrm{Cl}_3(A)$ is finite then the sequence of sets A_n, $n \in \omega$, is stabilized since some n.

We say that the operation Cl_3 *is generated* by operations Cl_1 and Cl_2, if $\mathrm{Cl}_3(A) = \bigcup\limits_{n\in\omega} A_n$ for any finite set $A \subseteq M_3$. Further we consider a closure operation Cl_3, generated by operations Cl_1 and Cl_2.

Let $(\mathbf{T}_2; \leqslant_2)$ be a self-sufficient class having the uniform t-amalgamation property and such that there exists an equivalence relation E on the universe of $(\mathbf{T}_3; \leqslant_3)$-generic model $\mathcal{M}_3$, satisfying the following conditions for any finite set $A \subseteq M_3$, where $\mathcal{M}_3 \models \Phi(A)$ for some type $\Phi(A) \in \mathbf{T}_3$:

1) $\mathrm{Cl}_1(A) = \bigcup\limits_{a\in A} \mathrm{Cl}_1(A \cap E(a))$;

2) if $B \subseteq \bigcup\limits_{a\in \mathrm{Cl}_2(A)} E(a)$ and $\mathrm{Cl}_2(B) \subseteq \bigcup\limits_{a\in \mathrm{Cl}_2(A)} E(a)$, then $\mathrm{Cl}_2(B) = B$;

3) there is a finite number m_A of E-classes $E_1, \ldots, E_{m_A}$, such that m_A is described by some formula from $\Phi(A)$, and $\mathrm{Cl}_3(A) \subseteq \bigcup\limits_{i=1}^{m_A} E_i$.

Then we say that $(\mathrm{Cl}_1, \mathrm{Cl}_2)$ is an *E-stepped special system of closures*, or an ESS-*system*.

THEOREM. *Let* $(\mathbf{T}_1; \leqslant_1)\mathcal{F}_{(\mathbf{T}_0;\leqslant_0)}(\mathbf{T}_2; \leqslant_2)$ *be a generic class,* $(\mathrm{Cl}_1, \mathrm{Cl}_2)$ *be an* ESS-*system, and* $(\mathbf{T}_i; \leqslant_i)$*,* $i = 1, 2$*, be classes having the finite closure property. Then the class* $(\mathbf{T}_1; \leqslant_1)\,\mathcal{F}_{(\mathbf{T}_0;\leqslant_0)}\,(\mathbf{T}_2; \leqslant_2)$ *has the finite closure property.*

Using theorem the author constructed fusions with ESS-systems realizing structural properties of stable Ehrenfeucht theories.

The theorem is naturally spreadable for *n-stepped* special systems $(\mathrm{Cl}_1, \ldots, \mathrm{Cl}_n)$ of closures.

[1] *Hrushovski E.* Strongly minimal expansions of algebraically closed fields // Israel J. Math. 1992. V. 79. P. 129–151.

[2] *Holland K.* An introduction to fusions of strongly minimal sets: The geometry of fusions // Archive for Math. Logic. 1995. V. 34. P. 395–413.

[3] *Holland K.* Strongly minimal fusions of vector spaces // Ann. Pure and Appl. Logic. 1997. V. 83, No. 1. P. 1–22.

[4] *Hasson A., Hils M.* Fusion over sublanguages // J. Symbolic Logic. 2006. V. 71, No. 2. P. 361–398.

[5] *Baudisch A., Martin-Pizarro A., Ziegler M.* Fusion over a vector space. — Preprint. Mathematisches Institut, Freiburg, 2005. — 22 p.

[6] *Baudisch A., Martin-Pizarro A., Ziegler M.* Hrushovski's Fusion. — Preprint. Mathematisches Institut, Freiburg, 2006. — 14 p.

[7] *Ziegler M.* Fusion of structures of finite Morley rank / Preprint. Mathematisches Institut, Freiburg, 2006. — 15 p.

[8] *Sudoplatov S. V.* Inessential combinations and colorings of models // Siberian Math. J. 2003. V. 44, No. 5. P. 1132–1141.

[9] *Sudoplatov S.V.* Complete theories with finitely many countable models. II // Algebra and Logic. 2006. V. 45, No. 3. P. 314–353.

[10] *Sudoplatov S.V.* Syntactic approach to construction of generic models // Algebra and Logic (submitted in 2005, available in: http://www.sudoplatov.fromru.com/papers/sppgmeng.pdf). — 23 p.

HTTP://WWW.SUDOPLATOV.FROMRU.COM/

E-mail: sudoplat@ngs.ru