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Abstracts

A two cardinal theorem for sets of types in stable theory

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Let L will be a first order finitary language. Structures will be denoted by A, B, etc. T will be a complete L-theory. A type of T over C is a set $\Delta(v_0)$ of $L(C)$-formulas which is consistent. A type over $\emptyset$ is called simply a type or a pure type. If $\Gamma(v_0)$ is a pure type and C a set then $\Gamma(C) = \{c \mid c \in C \text{ and } \models \Gamma(c)\}$.

A is λ-saturated if for every $C \subset A$ with $|C| < |A|$ and every $p \in S(C), p$ is realized by an element of A. A is λ-compact if every type $\Delta(v_0)$ over A with $|\Delta(v_0)| < \lambda$ is realized by an element of A. For $\lambda > |T|$ the two notions are equivalent and for $\lambda \leqslant |T|$ the notion of λ-compact is weaker than λ-saturated.

T is called stable in λ iff for all $C, |C| = \lambda$ implies $|S(C)| = \lambda$. T is stable if it is stable in some cardinal. For T stable, define $\mu(T)$ to be the first infinite cardinal μ such that T has no Morley tree of height μ.

Proposition 1.1 (Shelah) If T is stable then $\mu(T) \leqslant |T|^+$.

Proposition 1.2 If T is stable and $L' \subset L$ then $T' = T \restriction L'$, the reduct of T to L', is stable and $\mu(T') \leqslant \mu(T)$.

If $p \in S(C)$, we say that $\Delta(v_0)$ isolates p if $\Delta(v_0) \subset p$ and p is the unique extension of $\Delta(v_0)$ to a type in $S(C)$. p is λ-isolated if it is isolated by a type $\Delta(v_0)$ with $|\Delta(v_0)| < \lambda$.

Proposition 1.3. (Ressayre, Rowbottom, Shelah) If $\Delta(v_0) \subset F(C)$ is a type then there is $\Sigma(v_0)$ such that $\Sigma(v_0) \supset \Delta(v_0), |\Sigma(v_0) - \Delta(v_0)| < \mu(T)$ and $\Sigma(v_0)$ isolates a type of $S(C)$.

The most important consequence of 1.3 is the existence of λ-prime model $A \supset C$ (λ-compact and every elementary map of C into any λ-compact model B can be extended to an elemantary embedding of A into B).

Theorem 1.4. (Ressayre, Rowbottom, Shelah) If T is stable and $\lambda \geqslant \mu(T)$ then for every set C there is a model $A \models T$ which is λ-prime over C.

Definition 1.5. (Shelah) We say that a type $p \in S(C)$ does not split over $D \subset C$ if for every $\overline{a}, \overline{b} \in C$ and every L-formula $\psi(v_0, \overline{x})$, if $p(\overline{a}, D) = p(\overline{b}, D)$ then $\psi(v_0, \overline{a}) \in p$ iff $\psi(v_0, \overline{b}) \in p$.

Theorem 1.6. (Shelah, Harnik). If T is stable and $p \in S(C)$ then there is $D \subset C, |D| < \mu(T)$ such that p does not split over D.

Corollary 1.7. Let I be a set of indisernibles over C. If d is an element then there exists $I_0 \subset I, |I_0| < \mu(T)$, such that $I - I_0$ is a set of indiscernibles over $C \cup \{d\}$. Consequently, if D is a set of cardinality less than λ, where $\lambda > \mu(T)$ or $\lambda = \mu(T)$ and λ regular, then there is $I_0 \subset I, |I_0| < \lambda$ such that $I - I_0$ is a set of indiscernibles over $C \cup D$.

Proposition 1.8. Assume T to be stable. Given sets C and $\{d_\alpha\}_{\alpha<\kappa}$, denote $p_\alpha = p(d_\alpha, C \cup \{d_\beta\}_{\beta<\alpha})$. If $p_\beta \subset p_\alpha$ whenever $\beta < \alpha < \kappa$ and p_α does not split over C then $\{d_\alpha\}_{\alpha<\kappa}$ is a set of indiscernibles over C.

The principle captured by 1.8 originates with Morley (theorem 4.6), and has been used repeatedly by Shelah and Harnik.

The following beautiful definability theorem is due independently to Baldwin and Shelah, and was crucial in Lachlan's proof. It is, in a sense, a refinement of 1.6 obtained by "localizing" that statement to ψ-types. This idea of "localizing" comes from Shelah's and proved to be very fruitful.

Theorem 1.9. Assume T to be stable. If C is a set and a an element then for every L-formula $\psi(u, \overline{x})$ there is an $L(C)$-formula $\phi_\psi(\overline{x})$ such that for all $\overline{c} \in C, \models \psi(a, \overline{c})$ iff $\models \phi_\psi(\overline{c})$.

Theorem 2.2. Let T be a stable theory, λ a cardinal, $\lambda \geqslant \mu(T)$ and $\Gamma_i(v_0)$ a pure type with $|\Gamma_i(v_0)| < \lambda$. If C is λ-compact with respect to $\{\Gamma_i(v_0)\}_{i\leqslant\kappa}$, $\kappa < \lambda$ and A is a model of T which is λ-prime over C then $\Gamma_i(A) = \Gamma_i(C)$ for all $i \leqslant \kappa$ (in case $\kappa = 1$ it is the result of Harnik).

Lemma 2.3. If C is λ-compact with respect to $\{\Gamma_i(v_0)\}_{i\leqslant\kappa}$ and a realizes a λ-isolated type over C then $C \cup \{a\}$ is λ-compact with respect to $\Gamma_i(v_0)$ (also, $\Gamma_i(C \cup \{a\}) = \Gamma_i(C)$).

Lemma 2.4. If C is an increasing sequence of λ-compact sets with respect to $\{\Gamma_i(v_0)\}_{i\leqslant\kappa}$ and if $\Gamma_i(C_\alpha) = \Gamma_i(C_\delta)$ for all $\alpha < \delta$, then $C_\delta = \bigcup_{\alpha<\delta} C_\alpha$ is also λ-compact with respect to $\Gamma_i(v_0)$, $i \leqslant \kappa$.

We now turn to the main result of this report, is an analogue of Lachlan's and Harnik's results. The proof is inspired by Lachlan's and Harnik's. As a matter of fact, we have in mind a simplified version of that proof due to Shelah (Harnik). For a slightly different simplified proof see Baldwin.

Main lemma 3.1. Let T be a stable theory, $\lambda \geqslant \mu(T)$ and $\Gamma_i(v_0)$ a type of T with $|\Gamma_i(v_0)| < \lambda$. If T has a pair of λ-compact models A_0, A_1 such that $A_0 \prec A_1$ and $\Gamma_i(A_1) = \Gamma_i(A_0)$, $i \leqslant \kappa$, then there exists a λ-compact model A_2 such that $A_1 \prec A_2$ and $\Gamma_i(A_2) = \Gamma_i(A_0)$, $i \leqslant \kappa$.

Claim 3.2. $A_1 \cup \{c_1\}$ is λ-compact with respect to $\Gamma_i(v_0), i \leqslant \kappa$, where c_1 realizing q, does not split (definability by formulas for type c_0) over A_0 and $p(c_0, A_0) \subset q, c_0 \in A_1 - A_0$.

Corollary 3.3. If A_0, A_1 are λ-compact models of a stable theory T such that $A_0 \prec A_1$ and $\Gamma_i(A_1) = \Gamma_i(A_0)$, where $\lambda \geqslant \mu(T)$ and $\lambda > |\Gamma_i(v_0)|$, $i \leqslant \kappa$, then we can find λ-compact models A_α, $\alpha \geqslant 2$, such that $A_\beta \prec A_\alpha$ whenever $0 \leqslant \beta < \alpha$ and $\Gamma_i(A_\alpha) = \Gamma_i(A_0)$ for all α, $i \leqslant \kappa$. Thus, A_1 has arbitrarily large elementary extensions A with $\Gamma_i(A) = \Gamma_i(A_0)$, $i \leqslant \kappa$.

Lemma 4.1. If $A_0 \prec A_1$ are λ-compact models of a stable theory T, $\lambda \geqslant \mu(T)$ such that $\Gamma_i(A_1) = \Gamma_i(A_0)$, where $|\Gamma_i(v_0)| < \lambda$, $i \leqslant \kappa$, then ther exists a model A of T such that:

(i) $A_0 \prec A$ and $\Gamma_i(A) = \Gamma_i(A_0)$, $i \leqslant \kappa$;

(ii) there is a sequence $\{c_n\}, n < \omega, c_n \in A$ of indiscernibles over A such that for any $n < \omega, A_0 \cup \{c_i\}_{i<n}$ is included in a λ-compact elementary substructure of A.

Theorem 4.2. If A_0, A_1 are λ-compact models of a stable theory T such that $A_0 \prec A_1$ and $\Gamma_i(A_1) = \Gamma_i(A_0)$, $i \leqslant \kappa$, and if $\lambda, \lambda_1 \geqslant \mu(T), \lambda, \lambda_1 > |\Gamma_i(v_0)|$, then for every cardinal $k_1 > \lambda_1$, there are λ_1-compact models $B_0, B_1 \models T$ such that $B_0 \prec B_1$, $\Gamma_i(B_1) = \Gamma_i(B_0)$, $i \leqslant \kappa$, and $|B_0| \leqslant \lambda_1^{|T|}, |B_1| \geqslant k_1$.

Theorem 4.3. If A_0, A_1 satisfy the assumptions of 4.2 and, in adittion, $\Sigma_j(A_1) \neq \Sigma_j(A_0)$ for some types $\Sigma_j(v_0)$ then T has λ_1-compact models B_0, B_1 such that

$$B_0 \prec B_1, \Gamma_i(B_1) = \Gamma_i(B_0), i \leqslant \kappa, |B_0| \leqslant \lambda_1^T$$

and $|\Sigma_j(A_1)| \geqslant k_1$, for all j.

Theorem 5.1. (Shelah) If T is stable, $A_0, A_1 \models T, A_0 \prec A_1$ and, for a unary predicates $Q_i(v_0), Q_i(A_1) = Q_i(A_0), |Q_i(A_0)| \geqslant \aleph_0$ then for any $\kappa > \mu \geqslant |T|, T$ has a model B with $|B| = \kappa$ and $Q_i(B) = \mu$.

Lemma 6.1. If C is algebraically closed, $I_j(v_0) \subset F(C)$ a minimal type, $|I_j(v_0)| < \lambda$, $j \leqslant k$, and $|I_j(C)| \geqslant \lambda$ then C is λ-compact with respect to $I_j(v_0)$, $i \leqslant k$.

Theorem 6.2. If A_0, A_1 are λ-compact models of a stable theory T, $A_0 \prec A_1$ and $I_j(A_1) = I_j(A_0)$ where $I_j(v_0)$ is a minimal types with $|I_j(v_0)| < \lambda, \lambda_1$, $j \leqslant k < \lambda$, and if $\lambda, \lambda_1 \geqslant \mu(T), \lambda_1 \geqslant |T|$ then for every cardinal $k_1 > \lambda$, there is a λ_1-compact model $B_1 \models T$ such that $|B_1| = k_1$ and $|I_j(B_0)| = \lambda_1$ (thus, B_1 is not λ_1^+-compact for all $j \leqslant k$).

Proposition 6.3. (Harnik, Ressayre) If T is stable then exists a types $I_j(v_0)$ over a set C such that $I_j(v_0)$ is minimal and $|I_j(v_0)| \leqslant |T|$ (in fact, $|I_j(v_0)| < \mu(T)$), $j \leqslant k$.

Theorem 6.4. (Shelah) If a theory T has a model A with $|A| > \lambda^{|T|}$ which is $|T|^+$-saturated but not λ^+-saturated than for every regular cardinal λ_1, T has models of arbitrarily large cardinalities which are λ_1-saturated but not λ_1^+-saturated.

Theorem 6.5. (Harnik) If T is a stable and has a model A with $|A| > \lambda^{|T|}$ which is $|T|^+$-saturated but not λ^+-saturated than for every $\lambda_1 > |T|, T$ has models of arbitrary large cardinalities which are λ_1-saturated but not λ_1^+-saturated.

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