Survey of dualities for Nelson lattices

Sergei Odintsov[1]

[1]Sobolev Institute of Mathematics
odintsov@math.nsc.ru

MM'13, 13.11.13

Constructive properties of *Int*

If $\vdash_{Int} \varphi \vee \psi$, *then* $\vdash_{Int} \varphi$ *or* $\vdash_{Int} \psi$.

(Disjunctive property)

If $\vdash_{Int} \exists x \varphi(x)$, *there is a term* t *such that* $\vdash_{Int} \varphi(t)$

(Extracting terms from proofs)

Dual constructive properties, which fail for *Int*:

If $\vdash \sim (\varphi \wedge \psi)$, *then* $\vdash \sim \varphi$ *or* $\vdash \sim \psi$.

(Constructive negation property)

If $\vdash \sim \forall x \varphi(x)$, *there is a term* t *such that* $\vdash \sim \varphi(t)$.

(Effective counterexample property)

Constructive properties of *Int*

If $\vdash_{Int} \varphi \vee \psi$*, then* $\vdash_{Int} \varphi$ *or* $\vdash_{Int} \psi$*.*

(Disjunctive property)

If $\vdash_{Int} \exists x \varphi(x)$*, there is a term* t *such that* $\vdash_{Int} \varphi(t)$

(Extracting terms from proofs)

Dual constructive properties, which fail for *Int*:

If $\vdash \sim (\varphi \wedge \psi)$*, then* $\vdash \sim \varphi$ *or* $\vdash \sim \psi$*.*

(Constructive negation property)

If $\vdash \sim \forall x \varphi(x)$*, there is a term* t *such that* $\vdash \sim \varphi(t)$*.*

(Effective counterexample property)

Kleene realizability

- In [Nelson49], constructive arithmetic satisfying four constructive principles is defined via modification of Kleene realizability semantics
- Consider the language of arithmetic $\langle s^1, +^2, \cdot^2, 0\rangle$ and relation

$$e \circledR \varphi,$$

where $e \in N$ and φ is an arithmetic formula.

- $\langle\ ,\ \rangle : N \times N \to N$ is a primitive recursive 1-1 function.

Kleene realizability

- $e \circledR \bot$ is false for all e
- $e \circledR t = s$ iff $e = 0$ and $s = t$ is true
- $e \circledR \varphi \vee \psi$ iff $e = \langle n, m \rangle$ and $n = 0$, $m \circledR \varphi$, or $n > 0$, $m \circledR \psi$
- $e \circledR \varphi \wedge \psi$ iff $e = \langle n, m \rangle$, $n \circledR \varphi$ and $m \circledR \psi$
- $e \circledR \varphi \to \psi$ iff for any n, if $n \circledR \varphi$, then partial recursive function f_e is defined at n and $f_e(n) \circledR \psi$
- $e \circledR \forall x \varphi(x)$ iff for every n, $f_e(n) \circledR \varphi(\overline{n})$, where $\overline{n} = s^n(o)$
- $e \circledR \exists x \varphi(x)$ iff $e = \langle n, m \rangle$ and $m \circledR \varphi(\overline{n})$.

Kleene realizability

A formula φ is *realizable* if there is e such that $e \circledR \varphi$.

Theorem

(D. Nelson, 1947) *If φ is derivable in* **HA** *from realizable formulas, then φ is realizable.*

Markov principle.

$$\forall x(\varphi(x) \vee \neg\varphi(x)) \wedge \neg\forall x \neg\varphi(x) \to \exists x \varphi(x)$$

Markov principle is realizable, but non provable in **HA**.

Kleene realizability

A formula φ is *realizable* if there is e such that $e \circledR \varphi$.

Theorem

(D. Nelson, 1947) *If φ is derivable in* **HA** *from realizable formulas, then φ is realizable.*

Markov principle.

$$\forall x(\varphi(x) \vee \neg\varphi(x)) \wedge \neg\forall x \neg\varphi(x) \to \exists x \varphi(x)$$

Markov principle is realizable, but non provable in **HA**.

Kleene realizability

A formula φ is *realizable* if there is e such that $e \circledR \varphi$.

Theorem

(D. Nelson, 1947) *If φ is derivable in* **HA** *from realizable formulas, then φ is realizable.*

Markov principle.

$$\forall x(\varphi(x) \vee \neg\varphi(x)) \wedge \neg\forall x \neg\varphi(x) \to \exists x \varphi(x)$$

Markov principle is realizable, but non provable in **HA**.

Nelson's arithmetic

- Nelson introduced in the language additional negation symbol $\sim$ and defined two relations between natural numbers and arithmetical formulas

$$"e \circledR_p \varphi" \quad \text{and} \quad "e \circledR_n \varphi"$$

- p-Realizability is identical with Kleene realizability for old connectives and
- $e \circledR_p \sim \varphi$ iff $e \circledR_n \varphi$

Nelson's arithmetic

n-Realizability is defined as follows:

- $e \circledR_n \bot$ iff $e = 0$
- $e \circledR_n t = s$ iff $e = 0$ and $s = t$ is false
- $e \circledR_n \varphi \vee \psi$ iff $e = \langle k, m \rangle$, $k \circledR_n \varphi$ and $m \circledR_n \psi$
- $e \circledR_n \varphi \wedge \psi$ iff $e = \langle k, m \rangle$ and $k = 0$, $m \circledR_n \varphi$, or $k > 0$, $m\ n \circledR_n \psi$
- $e \circledR_n \varphi \to \psi$ iff $e = \langle k, m \rangle$, $k \circledR_p \varphi$ and $m \circledR_n \psi$
- $e \circledR_n \forall x \varphi(x)$ iff $e = \langle k, m \rangle$ and $m \circledR_n \varphi(\overline{k})$.
- $e \circledR_n \exists x \varphi(x)$ iff for every k, $f_e(k) \circledR_n \varphi(\overline{k})$
- $e \circledR_n \sim \varphi$ iff $e \circledR_p \varphi$

Nelson's arithmetic

The arithmetic **NA** suggested by D. Nelson satisfies the theorem

Theorem

(D. Nelson, 1947) *If* φ *is derivable in* **NA** *from p-realizable formulas, then* φ *is p-realizable.*

NA is a conservative extension of Heyting arithmetic **HA**

Nelson's arithmetic

NA has Disjunctive property and Extracting terms from proofs property and contains among others the following the axiom schemes

A1. $\sim\sim\varphi \leftrightarrow \varphi$

A2. $\sim(\varphi \vee \psi) \leftrightarrow (\sim\varphi \wedge \sim\psi)$

A3. $\sim(\varphi \wedge \psi) \leftrightarrow (\sim\varphi \vee \sim\psi)$

A4. $\sim(\varphi \to \psi) \leftrightarrow (\varphi \wedge \sim\psi)$

A5. $\sim\forall x\varphi(x) \leftrightarrow \exists x \sim\varphi(x)$

A6. $\sim\exists x\varphi(x) \leftrightarrow \forall x \sim\varphi(x)$

Nelson's arithmetic

Now we have

$\mathbf{NA} \vdash \sim (\varphi \wedge \psi)$ $\Rightarrow$*(A3)* $\mathbf{NA} \vdash \sim \varphi \vee \sim \psi$ $\Rightarrow$
$\Rightarrow$*(Disjunctive property)* $\mathbf{NA} \vdash \sim \varphi$ *or* $\mathbf{NA} \vdash \sim \psi$

and

$\mathbf{NA} \vdash \sim \forall x \varphi(x)$ $\Rightarrow$*(A5)* $\mathbf{NA} \vdash \exists x \sim \varphi(x)$ $\Rightarrow$
$\Rightarrow$*(Extracting terms)* $\mathbf{NA} \vdash \sim \varphi(t)$ *for some term t*

Historical remarks

- Propositional variant of Nelson logic was studied by N. Vorobiev (1952).
- A logic weaker then Nelson logic developed by Fitch (1952) it lacks $\sim(\varphi \to \psi) \leftrightarrow (\varphi \wedge \sim\psi)$.
- Algebraic semantics for propositional Nelson Logic was suggested by H.Rasiowa (1958).
- Kripke semantics for first order Nelson Logic was suggested by R.Thomason (1969).
- Independently, Gentzen-style calculus equivalent to Nelson's system was developed by F. von Kutschera (1969).
- The system closely related to strong negation systems arose also in the work by J.P. Cleave (1974), who constructed the predicate calculus adequate for the algebra of inexact sets of S. Körner (1966).

Historical remarks

- From the early 1970s several authors explored logical systems similar to Nelson's and Fitch's but lacking the “explosive" axiom $\sim\varphi \to (\varphi \to \psi)$, thus producing *paraconsistent* logics. The paraconsistent version of Nelson logic was studied independently by R. Routley (later R. Sylvan)(1974) in the propositional case, by Lopez-Escobar (1972) and by Nelson himself (1984), both in the first-order case.
- Nelson's logic was considered as logic suitable for description of information structures in monographs by H.Wansing (1993) and J.O.M.Jaspars (1994).
- H.Wansing applied strong negation to solve several well known paradoxes of philosophical logic.

Definition of basic logics

Logic is a set of formulas closed under the rules of *modus ponens* and *substitution*.

We consider two variants of Nelson's paraconsistent logic. The logic **N4** is determined in the language $\langle \vee, \wedge, \rightarrow, \sim \rangle$ with $\sim$ for strong negation by axioms:

1. Axiom of positive intuitionistic logic
2. Strong negation axioms (Vorobiev axioms):

   A1. $\sim\sim p \leftrightarrow p$
   A2. $\sim (p \vee q) \leftrightarrow (\sim p \wedge \sim q)$
   A3. $\sim (p \wedge q) \leftrightarrow (\sim p \vee \sim q)$
   A4. $\sim (p \rightarrow q) \leftrightarrow (p \wedge \sim q)$

Definition of basic logics

N4$^{\perp}$ is determined in the language $\langle \vee, \wedge, \rightarrow, \sim, \perp \rangle$ with additional symbol $\perp$ for absurdity by axioms of **N4** and

$$\text{A5. } \perp \rightarrow p \text{ and A6. } p \rightarrow \sim \perp,$$

in which case $\neg\varphi := \varphi \rightarrow \perp$ is an intuitionistic implication.

Theorem

The logic **N4**$^{\perp}$ *is a conservative extension of* **N4** *and of intuitionistic logic.*

$$\mathbf{N3} := \mathbf{N4} + \{\sim p \rightarrow (p \rightarrow q)\}$$

Put $\perp :=\sim (p \rightarrow p)$. Then

$$\mathbf{N3} \vdash \perp \rightarrow p, \ p \rightarrow \sim \perp.$$

Algebraic semantics for $\mathbf{N4}^{\perp}$

- [H. Rasiowa, 1958] The algebraic semantics for **N3** (with two negations $\sim$ and $\neg$) in terms of *N*-lattices (quasi-pseudo-Boolean algebras).
- [D. Vakarelov, 1977] and independently [M.M. Fidel, 1978] Presentation of *N*-lattices via *twist-structures* (the term is due to [M. Kracht, 1998]).
- [Odintsov, 2013] Algebraic semantics for **N4**

Algebraic semantics for **N4**$^{\perp}$

- [H. Rasiowa, 1958] The algebraic semantics for **N3** (with two negations $\sim$ and $\neg$) in terms of *N*-lattices (quasi-pseudo-Boolean algebras).
- [D. Vakarelov, 1977] and independently [M.M. Fidel, 1978] Presentation of *N*-lattices via *twist-structures* (the term is due to [M. Kracht, 1998]).
- [Odintsov, 2013] Algebraic semantics for **N4**

Algebraic semantics for $\mathbf{N4}^{\perp}$

- [H. Rasiowa, 1958] The algebraic semantics for **N3** (with two negations $\sim$ and $\neg$) in terms of *N*-lattices (quasi-pseudo-Boolean algebras).
- [D. Vakarelov, 1977] and independently [M.M. Fidel, 1978] Presentation of *N*-lattices via *twist-structures* (the term is due to [M. Kracht, 1998]).
- [Odintsov, 2013] Algebraic semantics for **N4**

Replacement of equivalents in Nelson's logics

- **N3**, **N4** and **N4**$^{\perp}$ are not closed under the replacement rule, but they are closed under the *weak replacement rule*:

$$\frac{\varphi \leftrightarrow \psi, \ \sim \varphi \leftrightarrow \sim \psi}{\xi(\varphi) \leftrightarrow \xi(\psi)}$$

- $\sim (p \to q) \leftrightarrow (p \wedge \sim q) \in$ **N4**, but $(p \to q) \leftrightarrow (\sim p \vee q) \in$ **N4**

- However, **N3**, **N4** and **N4**$^{\perp}$ a are closed under the *positive replacement rule*:

$$\frac{\varphi \leftrightarrow \psi}{\xi(\varphi) \leftrightarrow \xi(\psi)},$$

where ξ is $\sim$-free

Replacement of equivalents in Nelson's logics

- **N3**, **N4** and **N4**$^{\perp}$ are not closed under the replacement rule, but they are closed under the *weak replacement rule*:

$$\frac{\varphi \leftrightarrow \psi, \ \sim\varphi \leftrightarrow \sim\psi}{\xi(\varphi) \leftrightarrow \xi(\psi)}$$

- $\sim (p \to q) \leftrightarrow (p \wedge \sim q) \in$ **N4**, but $(p \to q) \leftrightarrow (\sim p \vee q) \in$ **N4**

- However, **N3**, **N4** and **N4**$^{\perp}$ a are closed under the *positive replacement rule*:

$$\frac{\varphi \leftrightarrow \psi}{\xi(\varphi) \leftrightarrow \xi(\psi)},$$

where ξ is $\sim$-free

Replacement of equivalents in Nelson's logics

- **N3**, **N4** and **N4**$^{\perp}$ are not closed under the replacement rule, but they are closed under the *weak replacement rule*:

$$\frac{\varphi \leftrightarrow \psi, \ \sim \varphi \leftrightarrow \sim \psi}{\xi(\varphi) \leftrightarrow \xi(\psi)}$$

- $\sim (p \to q) \leftrightarrow (p \wedge \sim q) \in$ **N4**, but $(p \to q) \leftrightarrow (\sim p \vee q) \in$ **N4**

- However, **N3**, **N4** and **N4**$^{\perp}$ a are closed under the *positive replacement rule*:

$$\frac{\varphi \leftrightarrow \psi}{\xi(\varphi) \leftrightarrow \xi(\psi)},$$

where ξ is $\sim$-free

Twist-structures

Let $\mathcal{A} = \langle A, \vee, \wedge, \to, 1\rangle(\langle A, \vee, \wedge, \to, 0, 1\rangle)$ be an implicative lattice (a Heyting algebra).

1. A *full twist-structure* over $\mathcal{A}$ is an algebra

$$\mathcal{A}^{\bowtie} = \langle A \times A, \vee, \wedge, \to, (\bot,) \sim\rangle$$

   with twist-operations:

$$(a, b) \vee (c, d) := (a \vee c, b \wedge d),\ (a, b) \wedge (c, d) := (a \wedge c, b \vee d)$$

$$(a, b) \to (c, d) := (a \to c, a \wedge d)$$

$$\sim (a, b) := (b, a),\ (\bot := (0, 1)).$$

2. A *twist-structure* over $\mathcal{A}$ is a subalgebra $\mathcal{B}$ of $\mathcal{A}^{\bowtie}$ such that $\pi_1(\mathcal{B}) = \mathcal{A}$
3. $S^{\bowtie}(\mathcal{A})$ is the class of all twist-structures over $\mathcal{A}$.

Twist-structures

Let $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$. For a formula φ, $\mathcal{B} \models^{\bowtie} \varphi$ if

$$\pi_1 v(\varphi) = 1$$

for any $\mathcal{B}$-valuation v.

$\models_{\bowtie} \varphi$ ($\models^{\perp}_{\bowtie} \varphi$) means that $\mathcal{B} \models^{\bowtie} \varphi$ for any twist-structure $\mathcal{B}$ over an implicative lattice (a Heyting algebra).

Theorem (Completeness)

$$\varphi \in \mathbf{N4} \;\Leftrightarrow \models_{\bowtie} \varphi;$$

$$\varphi \in \mathbf{N4}^{\perp} \;\Leftrightarrow \models^{\perp}_{\bowtie} \varphi$$

Invariants of twist-structures

- For an implicative lattice $\mathcal{A}$, denote

$$\mathcal{F}_d(\mathcal{A}) = \{a \vee (a \to b) \mid a \in \mathcal{A}\}$$

- If $\mathcal{A}$ is a Heyting algebra, then

$$\mathcal{F}_d(\mathcal{A}) = \{a \vee \neg a \mid a \in \mathcal{A}\}$$

- Let $\mathcal{A}$ be a Heyting algebra, ∇ be a filter on $\mathcal{A}$ such that $\mathcal{F}_d(\mathcal{A}) \subseteq \nabla$, and let Δ be an ideal on $\mathcal{A}$. Then there exists a twist-structure $Tw(\mathcal{A}, \nabla, \Delta) \in S^{\bowtie}(\mathcal{A})$ with the universe

$$|Tw(\mathcal{A}, \nabla, \Delta)| = \{(a, b) | \ a, b \in A, a \vee b \in \nabla, a \wedge b \in \Delta\}.$$

Invariants of twist-structures

- For an implicative lattice $\mathcal{A}$, denote

$$\mathcal{F}_d(\mathcal{A}) = \{a \vee (a \to b) \mid a \in \mathcal{A}\}$$

- If $\mathcal{A}$ is a Heyting algebra, then

$$\mathcal{F}_d(\mathcal{A}) = \{a \vee \neg a \mid a \in \mathcal{A}\}$$

- Let $\mathcal{A}$ be a Heyting algebra, ∇ be a filter on $\mathcal{A}$ such that $\mathcal{F}_d(\mathcal{A}) \subseteq \nabla$, and let Δ be an ideal on $\mathcal{A}$. Then there exists a twist-structure $Tw(\mathcal{A}, \nabla, \Delta) \in S^{\bowtie}(\mathcal{A})$ with the universe

$$|Tw(\mathcal{A}, \nabla, \Delta)| = \{(a, b) \mid a, b \in A, a \vee b \in \nabla, a \wedge b \in \Delta\}.$$

Invariants of twist-structures

- Let $\mathcal{A}$ be a Heyting algebra and $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$. We define

  $$\nabla(\mathcal{B}) := \{a \vee b |\ (a, b) \in \mathcal{B}\},\ \ \Delta(\mathcal{B}) := \{a \wedge b |\ (a, b) \in \mathcal{B}\}.$$

  Then $\mathcal{F}_d(\mathcal{A}) \subseteq \nabla(\mathcal{B})$ is a filter on $\mathcal{A}$ and $\Delta(\mathcal{B})$ is an ideal on $\mathcal{A}$. Moreover,

  $$\mathcal{B} = Tw(\mathcal{A}, \nabla(\mathcal{B}), \Delta(\mathcal{B})).$$

- Let $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$.

  $\mathcal{B} \models$ **N3** iff $\nabla(\mathcal{B}) = \{0\}$ iff $a \wedge b = 0$ for all $(a, b) \in \mathcal{B}$

- [Sendlewski, 1984]
  $\mathcal{B} \models$ **N3** iff $\mathcal{B} = Tw(\mathcal{A}, \nabla(\mathcal{B}), \{0\})$ and $\mathcal{F}_d(\mathcal{B}) \subseteq \nabla(\mathcal{B})$

Homomorphisms of twist-structures

- For $\mathcal{B}_i = Tw(\mathcal{A}_i, \nabla(\mathcal{B}_i), \Delta(\mathcal{B}_i))$, $i = 1, 2$ and $f : \mathcal{B}_1 \to \mathcal{B}_2$ is a homomorphism, then for $a \in \mathcal{A}_1$ we define

$$f'(a) = \pi_1 f((a, b)),$$

  where b is such that $(a, b) \in \mathcal{B}_1$.

- Then $f' : \mathcal{A}_1 \to \mathcal{A}_2$ is well defined and is a homomorphism. Moreover,

$$f'(\nabla(\mathcal{B}_1)) \subseteq \nabla(\mathcal{B}_2) \text{ and } f'(\Delta(\mathcal{B}_1)) \subseteq \Delta(\mathcal{B}_2).$$

Homomorphisms of twist-structures

- For $\mathcal{B}_i = Tw(\mathcal{A}_i, \nabla(\mathcal{B}_i), \Delta(\mathcal{B}_i))$, $i = 1, 2$ and $f : \mathcal{B}_1 \to \mathcal{B}_2$ is a homomorphism, then for $a \in \mathcal{A}_1$ we define

$$f'(a) = \pi_1 f((a, b)),$$

  where b is such that $(a, b) \in \mathcal{B}_1$.

- Then $f' : \mathcal{A}_1 \to \mathcal{A}_2$ is well defined and is a homomorphism. Moreover,

$$f'(\nabla(\mathcal{B}_1)) \subseteq \nabla(\mathcal{B}_2) \text{ and } f'(\Delta(\mathcal{B}_1)) \subseteq \Delta(\mathcal{B}_2).$$

Homomorphisms of twist-structures

- If $f : \mathcal{A}_1 \to \mathcal{A}_2$ is a homomorphism such that

$$f(\nabla(\mathcal{B}_1)) \subseteq \nabla(\mathcal{B}_2) \text{ and } f(\Delta(\mathcal{B}_1)) \subseteq \Delta(\mathcal{B}_2),$$

  and $h : \mathcal{B}_1 \to \mathcal{B}_2$ is such that $h' = f$, then

$$h((a, b)) = (f(a), f(b)).$$

- If $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$, then
  - $Con(\mathcal{B}) \cong Con(\mathcal{A})$;
  - $\mathcal{B}$ is subdirectly irreducible iff $\mathcal{A}$ is subdirectly irreducible.

Nelson lattices

$\mathcal{A} = \langle A, \vee, \wedge, \to, \sim, \rangle$ is an **N4**-lattice if:

1. $\langle A, \vee, \wedge, \sim \rangle$ is a De Morgan algebra;
2. $\preceq$, where $a \preceq b$ denotes $(a \to b) \to (a \to b) = a \to b$, is a preordering on $\mathcal{A}$;
3. $\approx := \preceq \cap \preceq^{-1}$ is a congruence wrt $\vee, \wedge, \to$ and $\mathcal{A}_{\bowtie} := \langle A, \vee, \wedge, \to \rangle / \approx$ is an implicative lattice;
4. $\sim (a \to b) \approx a \wedge \sim b$;
5. $a \leq b$ if and only if $a \preceq b$ and $\sim b \preceq \sim a$.

$\mathcal{A} \models \varphi$ iff $\varphi \to \varphi = \varphi$ is an identity on $\mathcal{A}$

Nelson lattices

$\mathcal{A} = \langle A, \vee, \wedge, \to, \sim, \rangle$ is an **N4-lattice** if:

1. $\langle A, \vee, \wedge, \sim \rangle$ is a De Morgan algebra;
2. $\preceq$, where $a \preceq b$ denotes $(a \to b) \to (a \to b) = a \to b$, is a preordering on $\mathcal{A}$;
3. $\approx := \preceq \cap \preceq^{-1}$ is a congruence wrt $\vee, \wedge, \to$ and $\mathcal{A}_{\bowtie} := \langle A, \vee, \wedge, \to \rangle / \approx$ is an implicative lattice;
4. $\sim (a \to b) \approx a \wedge \sim b$;
5. $a \leq b$ if and only if $a \preceq b$ and $\sim b \preceq \sim a$.

$\mathcal{A} \models \varphi$ iff $\varphi \to \varphi = \varphi$ is an identity on $\mathcal{A}$

Nelson lattices

$\mathcal{A} = \langle A, \vee, \wedge, \to, \sim, \rangle$ is an **N4**-lattice if:

1. $\langle A, \vee, \wedge, \sim \rangle$ is a De Morgan algebra;
2. $\preceq$, where $a \preceq b$ denotes $(a \to b) \to (a \to b) = a \to b$, is a preordering on $\mathcal{A}$;
3. $\approx := \preceq \cap \preceq^{-1}$ is a congruence wrt $\vee, \wedge, \to$ and $\mathcal{A}_{\bowtie} := \langle A, \vee, \wedge, \to \rangle / \approx$ is an implicative lattice;
4. $\sim (a \to b) \approx a \wedge \sim b$;
5. $a \leq b$ if and only if $a \preceq b$ and $\sim b \preceq \sim a$.

$\mathcal{A} \models \varphi$ iff $\varphi \to \varphi = \varphi$ is an identity on $\mathcal{A}$

Nelson lattices

$\mathcal{A} = \langle A, \vee, \wedge, \to, \sim, \rangle$ is an **N4**-lattice if:

1. $\langle A, \vee, \wedge, \sim \rangle$ is a De Morgan algebra;
2. $\preceq$, where $a \preceq b$ denotes $(a \to b) \to (a \to b) = a \to b$, is a preordering on $\mathcal{A}$;
3. $\approx := \preceq \cap \preceq^{-1}$ is a congruence wrt $\vee, \wedge, \to$ and $\mathcal{A}_{\bowtie} := \langle A, \vee, \wedge, \to \rangle / \approx$ is an implicative lattice;
4. $\sim (a \to b) \approx a \wedge \sim b$;
5. $a \leq b$ if and only if $a \preceq b$ and $\sim b \preceq \sim a$.

$\mathcal{A} \models \varphi$ iff $\varphi \to \varphi = \varphi$ is an identity on $\mathcal{A}$

Nelson lattices

$\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, \sim, \rangle$ is an **N4**-lattice if:

1. $\langle A, \vee, \wedge, \sim \rangle$ is a De Morgan algebra;
2. $\preceq$, where $a \preceq b$ denotes $(a \rightarrow b) \rightarrow (a \rightarrow b) = a \rightarrow b$, is a preordering on $\mathcal{A}$;
3. $\approx := \preceq \cap \preceq^{-1}$ is a congruence wrt $\vee, \wedge, \rightarrow$ and $\mathcal{A}_{\bowtie} := \langle A, \vee, \wedge, \rightarrow \rangle / \approx$ is an implicative lattice;
4. $\sim (a \rightarrow b) \approx a \wedge \sim b$;
5. $a \leq b$ if and only if $a \preceq b$ and $\sim b \preceq \sim a$.

$\mathcal{A} \models \varphi$ iff $\varphi \rightarrow \varphi = \varphi$ is an identity on $\mathcal{A}$

Nelson lattices

$\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, \sim, \rangle$ is an **N4**-lattice if:

1. $\langle A, \vee, \wedge, \sim\rangle$ is a De Morgan algebra;
2. $\preceq$, where $a \preceq b$ denotes $(a \rightarrow b) \rightarrow (a \rightarrow b) = a \rightarrow b$, is a preordering on $\mathcal{A}$;
3. $\approx := \preceq \cap \preceq^{-1}$ is a congruence wrt $\vee, \wedge, \rightarrow$ and $\mathcal{A}_{\bowtie} := \langle A, \vee, \wedge, \rightarrow\rangle / \approx$ is an implicative lattice;
4. $\sim (a \rightarrow b) \approx a \wedge \sim b$;
5. $a \leq b$ if and only if $a \preceq b$ and $\sim b \preceq \sim a$.

$\mathcal{A} \models \varphi$ iff $\varphi \rightarrow \varphi = \varphi$ is an identity on $\mathcal{A}$

Nelson lattices

$\mathcal{A} = \langle A, \vee, \wedge, \to, \sim, \rangle$ is an **N4**-lattice if:

1. $\langle A, \vee, \wedge, \sim \rangle$ is a De Morgan algebra;
2. $\preceq$, where $a \preceq b$ denotes $(a \to b) \to (a \to b) = a \to b$, is a preordering on $\mathcal{A}$;
3. $\approx := \preceq \cap \preceq^{-1}$ is a congruence wrt $\vee, \wedge, \to$ and $\mathcal{A}_{\bowtie} := \langle A, \vee, \wedge, \to \rangle / \approx$ is an implicative lattice;
4. $\sim (a \to b) \approx a \wedge \sim b$;
5. $a \leq b$ if and only if $a \preceq b$ and $\sim b \preceq \sim a$.

$\mathcal{A} \models \varphi$ iff $\varphi \to \varphi = \varphi$ is an identity on $\mathcal{A}$

Nelson lattices

1. $\mathcal{A} = \langle A, \vee, \wedge, \to, \sim, \bot, 1 \rangle$ is a bounded **N4**-lattice if $\mathcal{A} = \langle A, \vee, \wedge, \to, \sim \rangle$ is an **N4**-lattice and $\bot$, 1 are the least and the greatest elements
2. If $\mathcal{A}$ is an implicative lattice (a Heyting algebra) and $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$, then $\mathcal{B}$ is a (bounded) **N4**-lattice
3. If $\mathcal{B}$ is an **N4**-lattice, then

$$h(a) = ([a]_{\approx}, [\sim a]_{\approx})$$

embeds $\mathcal{B}$ into $(\mathcal{B}_{\bowtie})^{\bowtie}$

Nelson lattices

1. $\mathcal{A} = \langle A, \vee, \wedge, \to, \sim, \bot, 1\rangle$ is a bounded **N4**-lattice if $\mathcal{A} = \langle A, \vee, \wedge, \to, \sim\rangle$ is an **N4**-lattice and $\bot$, 1 are the least and the greatest elements

2. If $\mathcal{A}$ is an implicative lattice (a Heyting algebra) and $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$, then $\mathcal{B}$ is a (bounded) **N4**-lattice

3. If $\mathcal{B}$ is an **N4**-lattice, then

$$h(a) = ([a]_{\approx}, [\sim a]_{\approx})$$

embeds $\mathcal{B}$ into $(\mathcal{B}_{\bowtie})^{\bowtie}$

Nelson lattices

1. $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, \sim, \bot, 1\rangle$ is a **bounded N4-lattice** if $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, \sim\rangle$ is an **N4**-lattice and $\bot$, 1 are the least and the greatest elements
2. If $\mathcal{A}$ is an implicative lattice (a Heyting algebra) and $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$, then $\mathcal{B}$ is a (bounded) **N4**-lattice
3. If $\mathcal{B}$ is an **N4**-lattice, then

$$h(a) = ([a]_{\approx}, [\sim a]_{\approx})$$

embeds $\mathcal{B}$ into $(\mathcal{B}_{\bowtie})^{\bowtie}$

Nelson lattices

1. $\mathcal{A} = \langle A, \vee, \wedge, \to, \sim, \perp, 1\rangle$ is a **bounded N4-lattice** if $\mathcal{A} = \langle A, \vee, \wedge, \to, \sim\rangle$ is an **N4**-lattice and $\perp$, 1 are the least and the greatest elements
2. If $\mathcal{A}$ is an implicative lattice (a Heyting algebra) and $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$, then $\mathcal{B}$ is a (bounded) **N4**-lattice
3. If $\mathcal{B}$ is an **N4**-lattice, then

$$h(a) = ([a]_{\approx}, [\sim a]_{\approx})$$

embeds $\mathcal{B}$ into $(\mathcal{B}_{\bowtie})^{\bowtie}$

Nelson lattices

1. $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, \sim, \bot, 1\rangle$ is a bounded **N4**-lattice if $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, \sim\rangle$ is an **N4**-lattice and $\bot$, 1 are the least and the greatest elements

2. If $\mathcal{A}$ is an implicative lattice (a Heyting algebra) and $\mathcal{B} \in S^{\bowtie}(\mathcal{A})$, then $\mathcal{B}$ is a (bounded) **N4**-lattice

3. If $\mathcal{B}$ is an **N4**-lattice, then

$$h(a) = ([a]_{\approx}, [\sim a]_{\approx})$$

embeds $\mathcal{B}$ into $(\mathcal{B}_{\bowtie})^{\bowtie}$

Nelson lattices

1. Let $(a, b) \in \mathcal{B} \in S^{\bowtie}(\mathcal{A})$.

$$(a, b) \to (a, b) = (1, a \wedge b)$$

Thus,

$$(a, b) \to (a, b) = (a, b) \text{ iff } a = 1$$

2. In this way,
$\mathcal{B} \models_{\bowtie} \varphi$ iff $\varphi \to \varphi = \varphi$ is an identity on $\mathcal{B}$

3. **N4** (**N4**$^{\perp}$) is complete wrt the class of (bounded) **N4**-lattices

Nelson lattices

1. Let $(a, b) \in \mathcal{B} \in S^{\bowtie}(\mathcal{A})$.

$$(a, b) \to (a, b) = (1, a \wedge b)$$

   Thus,

$$(a, b) \to (a, b) = (a, b) \text{ iff } a = 1$$

2. In this way,
   $\mathcal{B} \models_{\bowtie} \varphi$ iff $\varphi \to \varphi = \varphi$ is an identity on $\mathcal{B}$

3. **N4** (**N4**$^{\perp}$) is complete wrt the class of (bounded) **N4**-lattices

Nelson lattices

1. Let $(a, b) \in \mathcal{B} \in S^{\bowtie}(\mathcal{A})$.

$$(a, b) \to (a, b) = (1, a \wedge b)$$

   Thus,

$$(a, b) \to (a, b) = (a, b) \text{ iff } a = 1$$

2. In this way,
   $\mathcal{B} \models_{\bowtie} \varphi$ iff $\varphi \to \varphi = \varphi$ is an identity on $\mathcal{B}$

3. **N4** (**N4**$^{\perp}$) is complete wrt the class of (bounded) **N4**-lattices

Nelson lattices

1. Let $(a, b) \in \mathcal{B} \in S^{\bowtie}(\mathcal{A})$.

$$(a, b) \to (a, b) = (1, a \wedge b)$$

   Thus,

$$(a, b) \to (a, b) = (a, b) \text{ iff } a = 1$$

2. In this way,
   $\mathcal{B} \models_{\bowtie} \varphi$ iff $\varphi \to \varphi = \varphi$ is an identity on $\mathcal{B}$

3. **N4** (**N4**$^{\perp}$) is complete wrt the class of (bounded) **N4**-lattices

Nelson lattices

1. Let $(a, b) \in \mathcal{B} \in S^{\bowtie}(\mathcal{A})$.

$$(a, b) \to (a, b) = (1, a \wedge b)$$

   Thus,

$$(a, b) \to (a, b) = (a, b) \text{ iff } a = 1$$

2. In this way,
   $\mathcal{B} \models_{\bowtie} \varphi$ iff $\varphi \to \varphi = \varphi$ is an identity on $\mathcal{B}$
3. **N4** (**N4**$^{\perp}$) is complete wrt the class of (bounded) **N4**-lattices

Topological semantics: historical remarks

- [M. Stone, 1936]
  Duality of Boolean algebras and Stone spaces
  (compact totally disconnected spaces)
- [M. Stone 1937]
  "Less satisfactory" duality for distributive lattices
- [H. Priestley 1970]
  Duality of distributive lattices and Priestley spaces
  (compact totally order-disconnected spaces)
- [R. Cignoli 1986, A. Sendlewski 1990]
  Priestley duality for **N3**-Lattices (models of explosive Nelson's logic **N3**) based on interpolation property [Monteiro 1963]

Topological semantics: historical remarks

- [M. Stone, 1936]
  Duality of Boolean algebras and Stone spaces
  (compact totally disconnected spaces)
- [M. Stone 1937]
  "Less satisfactory" duality for distributive lattices
- [H. Priestley 1970]
  Duality of distributive lattices and Priestley spaces
  (compact totally order-disconnected spaces)
- [R. Cignoli 1986, A. Sendlewski 1990]
  Priestley duality for **N3**-Lattices (models of explosive Nelson's logic **N3**) based on interpolation property [Monteiro 1963]

Topological semantics: historical remarks

- [M. Stone, 1936]
  Duality of Boolean algebras and Stone spaces
  (compact totally disconnected spaces)
- [M. Stone 1937]
  "Less satisfactory" duality for distributive lattices
- [H. Priestley 1970]
  Duality of distributive lattices and Priestley spaces
  (compact totally order-disconnected spaces)
- [R. Cignoli 1986, A. Sendlewski 1990]
  Priestley duality for **N3**-Lattices (models of explosive Nelson's logic **N3**) based on interpolation property [Monteiro 1963]

Topological semantics: historical remarks

- [M. Stone, 1936]
  Duality of Boolean algebras and Stone spaces
  (compact totally disconnected spaces)
- [M. Stone 1937]
  "Less satisfactory" duality for distributive lattices
- [H. Priestley 1970]
  Duality of distributive lattices and Priestley spaces
  (compact totally order-disconnected spaces)
- [R. Cignoli 1986, A. Sendlewski 1990]
  Priestley duality for **N3**-Lattices (models of explosive Nelson's logic **N3**) based on interpolation property [Monteiro 1963]

Priestley Spaces

- $\mathcal{X} = (X, \leq, \tau)$ is an ordered topological space if $\leq$ is a partial order on X and τ a topology on X.
- $\mathcal{X}$ is a totally order-disconnected topological space if for any $x, y \in X$ with $x \not\leq y$, there is a clopen cone U such that $x \in U$ and $y \notin U$.
- Priestley space is a compact totally order-disconnected topological space.
- $\mathcal{T}$ is the category of Priestley spaces and order preserving continuous functions.

Priestley Spaces

- $\mathcal{X} = (X, \leq, \tau)$ is an ordered topological space if $\leq$ is a partial order on X and τ a topology on X.
- $\mathcal{X}$ is a totally order-disconnected topological space if for any $x, y \in X$ with $x \not\leq y$, there is a clopen cone U such that $x \in U$ and $y \notin U$.
- Priestley space is a compact totally order-disconnected topological space.
- $\mathcal{T}$ is the category of Priestley spaces and order preserving continuous functions.

Priestley Spaces

- $\mathcal{X} = (X, \leq, \tau)$ is an ordered topological space if $\leq$ is a partial order on X and τ a topology on X.
- $\mathcal{X}$ is a totally order-disconnected topological space if for any $x, y \in X$ with $x \not\leq y$, there is a clopen cone U such that $x \in U$ and $y \notin U$.
- Priestley space is a compact totally order-disconnected topological space.
- $\mathcal{T}$ is the category of Priestley spaces and order preserving continuous functions.

Priestley Spaces

- $\mathcal{X} = (X, \leq, \tau)$ is an ordered topological space if $\leq$ is a partial order on X and τ a topology on X.
- $\mathcal{X}$ is a totally order-disconnected topological space if for any $x, y \in X$ with $x \not\leq y$, there is a clopen cone U such that $x \in U$ and $y \notin U$.
- Priestley space is a compact totally order-disconnected topological space.
- $\mathcal{T}$ is the category of Priestley spaces and order preserving continuous functions.

- $\mathcal{D}$ is the category of bounded distributive lattices and their homomorphisms.
- For $\mathcal{A} \in Ob(\mathcal{D})$ define

$$\mathrm{X}(\mathcal{A}) = (X, \subseteq, \tau), \quad \text{where}$$

1. X is the set of prime filters on $\mathcal{A}$;
2. $\subseteq$ is the set inclusion;
3. τ is given by the subbase:

$$\sigma_{\mathcal{A}}(a) := \{P \in X \mid a \in P\} \text{ and } X \setminus \sigma_{\mathcal{A}}(a), \text{ where } a \in \mathcal{A}$$

- For $f : \mathcal{A} \to \mathcal{B}$, define $\mathrm{X}(f) : \mathrm{X}(\mathcal{B}) \to \mathrm{X}(\mathcal{A})$ by:

$$\mathrm{X}(f)(P) := f^{-1}(P)$$

Distributive lattices $\to$ Priestley spaces

- $\mathcal{D}$ is the category of bounded distributive lattices and their homomorphisms.
- For $\mathcal{A} \in Ob(\mathcal{D})$ define

$$\mathrm{X}(\mathcal{A}) = (X, \subseteq, \tau), \quad \text{where}$$

1. X is the set of prime filters on $\mathcal{A}$;
2. $\subseteq$ is the set inclusion;
3. τ is given by the subbase:

$$\sigma_{\mathcal{A}}(a) := \{P \in X \mid a \in P\} \text{ and } X \setminus \sigma_{\mathcal{A}}(a), \text{ where } a \in \mathcal{A}$$

- For $f : \mathcal{A} \to \mathcal{B}$, define $\mathrm{X}(f) : \mathrm{X}(\mathcal{B}) \to \mathrm{X}(\mathcal{A})$ by:

$$\mathrm{X}(f)(P) := f^{-1}(P)$$

Distributive lattices → Priestley spaces

- $\mathcal{D}$ is the category of bounded distributive lattices and their homomorphisms.
- For $\mathcal{A} \in Ob(\mathcal{D})$ define

$$\mathsf{X}(\mathcal{A}) = (X, \subseteq, \tau), \quad \text{where}$$

1. X is the set of prime filters on $\mathcal{A}$;
2. $\subseteq$ is the set inclusion;
3. τ is given by the subbase:

$$\sigma_{\mathcal{A}}(a) := \{P \in X \mid a \in P\} \text{ and } X \setminus \sigma_{\mathcal{A}}(a), \text{ where } a \in \mathcal{A}$$

- For $f : \mathcal{A} \to \mathcal{B}$, define $\mathsf{X}(f) : \mathsf{X}(\mathcal{B}) \to \mathsf{X}(\mathcal{A})$ by:

$$\mathsf{X}(f)(P) := f^{-1}(P)$$

Priestley spaces → Distributive lattices

- For $\mathcal{X} \in Ob(\mathcal{T})$, $\mathrm{D}(\mathcal{X})$ is the lattice of clopen cones in $\mathcal{X}$.
- For a $\mathcal{T}$-morphism $f : \mathcal{X} \to \mathcal{X}'$,

$$\mathrm{D}(f) : \mathrm{D}(\mathcal{X}') \to \mathrm{D}(\mathcal{X})$$

is given by the prescription $\mathrm{D}(f)(U) := f^{-1}(U)$.

Priestley spaces $\to$ Distributive lattices

- For $\mathcal{X} \in Ob(\mathcal{T})$, $\mathsf{D}(\mathcal{X})$ is the lattice of clopen cones in $\mathcal{X}$.
- For a $\mathcal{T}$-morphism $f : \mathcal{X} \to \mathcal{X}'$,

$$\mathsf{D}(f) : \mathsf{D}(\mathcal{X}') \to \mathsf{D}(\mathcal{X})$$

is given by the prescription $\mathsf{D}(f)(U) := f^{-1}(U)$.

Priestley Duality

- $\mathsf{X} : \mathcal{D} \to \mathcal{T}$ and $\mathsf{D} : \mathcal{T} \to \mathcal{D}$ are contravariant functors.
- For $\mathcal{X} \in Ob(\mathcal{T})$, $\varepsilon_{\mathcal{X}} : \mathcal{X} \to \mathsf{XD}(\mathcal{X})$ is a $\mathcal{T}$-isomorphism.

$$\varepsilon_{\mathcal{X}}(x) := \{U \in \mathsf{D}(\mathcal{X}) \mid x \in U\}$$

- For $\mathcal{A} \in Ob(\mathcal{D})$, $\sigma_{\mathcal{A}} : \mathcal{A} \to \mathsf{DX}(\mathcal{A})$ is a lattice isomorphism.

$$\sigma_{\mathcal{A}}(a) := \{P \in X \mid a \in P\}$$

Priestley Duality

- $\mathsf{X} : \mathcal{D} \to \mathcal{T}$ and $\mathsf{D} : \mathcal{T} \to \mathcal{D}$ are contravariant functors.
- For $\mathcal{X} \in Ob(\mathcal{T})$, $\varepsilon_{\mathcal{X}} : \mathcal{X} \to \mathsf{XD}(\mathcal{X})$ is a $\mathcal{T}$-isomorphism.

$$\varepsilon_{\mathcal{X}}(x) := \{U \in \mathsf{D}(\mathcal{X}) \mid x \in U\}$$

- For $\mathcal{A} \in Ob(\mathcal{D})$, $\sigma_{\mathcal{A}} : \mathcal{A} \to \mathsf{DX}(\mathcal{A})$ is a lattice isomorphism.

$$\sigma_{\mathcal{A}}(a) := \{P \in X \mid a \in P\}$$

Priestley Duality

- $\mathsf{X} : \mathcal{D} \to \mathcal{T}$ and $\mathsf{D} : \mathcal{T} \to \mathcal{D}$ are contravariant functors.
- For $\mathcal{X} \in Ob(\mathcal{T})$, $\varepsilon_{\mathcal{X}} : \mathcal{X} \to \mathsf{XD}(\mathcal{X})$ is a $\mathcal{T}$-isomorphism.

$$\varepsilon_{\mathcal{X}}(x) := \{U \in \mathsf{D}(\mathcal{X}) \mid x \in U\}$$

- For $\mathcal{A} \in Ob(\mathcal{D})$, $\sigma_{\mathcal{A}} : \mathcal{A} \to \mathsf{DX}(\mathcal{A})$ is a lattice isomorphism.

$$\sigma_{\mathcal{A}}(a) := \{P \in X \mid a \in P\}$$

Priestley duality

- $\sigma : \mathbf{1}_{\mathcal{D}} \cong \mathrm{D} \circ \mathrm{X}$ and $\varepsilon : \mathbf{1}_{\mathcal{T}} \cong \mathrm{X} \circ \mathrm{D}$ are natural isomorphisms, i.e., for any $\mathcal{D}$-morphism $f : \mathcal{A} \to \mathcal{B}$ and $\mathcal{T}$-morphism $g : \mathcal{X} \to \mathcal{Y}$, the diagrams below are commutative.

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{\sigma_{\mathcal{A}}} & \mathrm{DX}(\mathcal{A}) \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{DX}(f)} \\
\mathcal{B} & \xrightarrow{\sigma_{\mathcal{B}}} & \mathrm{DX}(\mathcal{B})
\end{array}
\qquad\qquad
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\varepsilon_{\mathcal{X}}} & \mathrm{XD}(\mathcal{X}) \\
{\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{XD}(f)} \\
\mathcal{Y} & \xrightarrow{\varepsilon_{\mathcal{Y}}} & \mathrm{XD}(\mathcal{Y})
\end{array}
$$

- Thus, the categories $\mathcal{D}$ and $\mathcal{T}$ are dually equivalent via functors X and D.

Priestley duality

- $\sigma : \mathbf{1}_{\mathcal{D}} \cong \mathrm{D} \circ \mathrm{X}$ and $\varepsilon : \mathbf{1}_{\mathcal{T}} \cong \mathrm{X} \circ \mathrm{D}$ are natural isomorphisms, i.e., for any $\mathcal{D}$-morphism $f : \mathcal{A} \to \mathcal{B}$ and $\mathcal{T}$-morphism $g : \mathcal{X} \to \mathcal{Y}$, the diagrams below are commutative.

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\sigma_{\mathcal{A}}} & \mathrm{DX}(\mathcal{A}) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \mathrm{DX}(f)} \\ \mathcal{B} & \xrightarrow{\sigma_{\mathcal{B}}} & \mathrm{DX}(\mathcal{B}) \end{array} \qquad \begin{array}{ccc} \mathcal{X} & \xrightarrow{\varepsilon_{\mathcal{X}}} & \mathrm{XD}(\mathcal{X}) \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \mathrm{XD}(f)} \\ \mathcal{Y} & \xrightarrow{\varepsilon_{\mathcal{Y}}} & \mathrm{XD}(\mathcal{Y}) \end{array}$$

- Thus, the categories $\mathcal{D}$ and $\mathcal{T}$ are dually equivalent via functors X and D.

Duality for Heyting algebras [Esakia 1974]

- Heyting space is a Priestley space $\mathcal{X}$ s. t. $U\downarrow$ is open for any U open in $\mathcal{X}$. ($U\downarrow = \{y \mid y \leq x$ for some $x \in U\}$)
- For a Heyting space $\mathcal{X}$, $\mathrm{D}_H(\mathcal{X})$ is the lattice $\mathrm{D}(\mathcal{X})$ with the operation $\supset$:

$$U \supset V := X \setminus (U \setminus V)\downarrow \text{ for any } U, V \in \mathrm{D}(\mathcal{X}).$$

  $\mathrm{D}_H(\mathcal{X})$ is a Heyting algebra.
- Equivalently,

$$U \supset V = \{x \in X \mid \forall y \geq x (y \in U \Rightarrow y \in V)\}$$

Duality for Heyting algebras [Esakia 1974]

- Heyting space is a Priestley space $\mathcal{X}$ s. t. $U\downarrow$ is open for any U open in $\mathcal{X}$. ($U\downarrow = \{y \mid y \leq x \text{ for some } x \in U\}$)
- For a Heyting space $\mathcal{X}$, $\mathrm{D}_H(\mathcal{X})$ is the lattice $\mathrm{D}(\mathcal{X})$ with the operation $\supset$:

$$U \supset V := X \setminus (U \setminus V)\downarrow \text{ for any } U, V \in \mathrm{D}(\mathcal{X}).$$

  $\mathrm{D}_H(\mathcal{X})$ is a Heyting algebra.
- Equivalently,

$$U \supset V = \{x \in X \mid \forall y \geq x(y \in U \Rightarrow y \in V)\}$$

Duality for Heyting algebras [Esakia 1974]

- Heyting space is a Priestley space $\mathcal{X}$ s. t. $U\downarrow$ is open for any U open in $\mathcal{X}$. ($U\downarrow = \{y \mid y \leq x$ for some $x \in U\}$)
- For a Heyting space $\mathcal{X}$, $\mathrm{D}_H(\mathcal{X})$ is the lattice $\mathrm{D}(\mathcal{X})$ with the operation $\supset$:

$$U \supset V := X \setminus (U \setminus V)\downarrow \text{ for any } U, V \in \mathrm{D}(\mathcal{X}).$$

  $\mathrm{D}_H(\mathcal{X})$ is a Heyting algebra.
- Equivalently,

$$U \supset V = \{x \in X \mid \forall y \geq x (y \in U \Rightarrow y \in V)\}$$

Duality for Heyting algebras [Esakia 1974]

- $\mathcal{T}$-morphism $f : \mathcal{X} \to \mathcal{Y}$ is a Heyting function if
$f^{-1}(U\downarrow) = (f^{-1}(U))\downarrow$ for any U open in $\mathcal{Y}$.

- Equivalently, Heyting function is a $\mathcal{T}$-morphism s.t. for all $x \in X$,

$$f(x)\uparrow \subseteq f(x\uparrow),$$

$$\text{or, } \forall y \geq_{\mathcal{Y}} f(x) \exists z \geq_{\mathcal{X}} x (f(z) = y)$$

In fact, Heyting functions are continuous p-morphisms [Maximova, 1972]

- Categories $\mathcal{H}^*$ of Heyting spaces and functions and $\mathcal{H}$ of Heyting algebras and their homomorphisms are dually equivalent via X_H and D_H.

Duality for Heyting algebras [Esakia 1974]

- $\mathcal{T}$-morphism $f : \mathcal{X} \to \mathcal{Y}$ is a Heyting function if
  $f^{-1}(U\downarrow) = (f^{-1}(U))\downarrow$ for any U open in $\mathcal{Y}$.
- Equivalently, Heyting function is a $\mathcal{T}$-morphism s.t. for all $x \in X$,

$$f(x)\uparrow \subseteq f(x\uparrow),$$

or, $\forall y \geq_{\mathcal{Y}} f(x) \exists z \geq_{\mathcal{X}} x (f(z) = y)$

In fact, Heyting functions are continuous p-morphisms [Maximova, 1972]

- Categories $\mathcal{H}^*$ of Heyting spaces and functions and $\mathcal{H}$ of Heyting algebras and their homomorphisms are dually equivalent via X_H and D_H.

Duality for Heyting algebras [Esakia 1974]

- $\mathcal{T}$-morphism $f : \mathcal{X} \to \mathcal{Y}$ is a Heyting function if
  $f^{-1}(U\downarrow) = (f^{-1}(U))\downarrow$ for any U open in $\mathcal{Y}$.
- Equivalently, Heyting function is a $\mathcal{T}$-morphism s.t. for all $x \in X$,

$$f(x)\uparrow \subseteq f(x\uparrow),$$

$$\text{or, } \forall y \geq_{\mathcal{Y}} f(x) \exists z \geq_{\mathcal{X}} x (f(z) = y)$$

  In fact, Heyting functions are continuous p-morphisms [Maximova, 1972]
- Categories $\mathcal{H}^*$ of Heyting spaces and functions and $\mathcal{H}$ of Heyting algebras and their homomorphisms are dually equivalent via X_H and D_H.

Duality for Heyting algebras [Esakia 1974]

- $\mathcal{T}$-morphism $f : \mathcal{X} \to \mathcal{Y}$ is a Heyting function if
  $f^{-1}(U\downarrow) = (f^{-1}(U))\downarrow$ for any U open in $\mathcal{Y}$.
- Equivalently, Heyting function is a $\mathcal{T}$-morphism s.t. for all $x \in X$,

$$f(x)\uparrow \subseteq f(x\uparrow),$$

$$\text{or, } \forall y \geq_{\mathcal{Y}} f(x) \exists z \geq_{\mathcal{X}} x (f(z) = y)$$

  In fact, Heyting functions are continuous p-morphisms [Maximova, 1972]

- Categories $\mathcal{H}^*$ of Heyting spaces and functions and $\mathcal{H}$ of Heyting algebras and their homomorphisms are dually equivalent via X_H and D_H.

Priestley duality for De Morgan algebras [Cornish & Fowler 77]

- A De Morgan space is a pair $(\mathcal{X}, g)$, where
  - $\mathcal{X}$ is a Priestley space;
  - $g : \mathcal{X} \to \mathcal{X}$ is an antimonotonic homeomorphism s.t. $g^2 = id_{\mathcal{X}}$.
- $\mathcal{T}$-morphism $f : (\mathcal{X}, g) \to (\mathcal{X}', g')$ is a De Morgan function if $fg = g'f$.
- For a De Morgan space $(\mathcal{X}, g)$, put:

$$\mathrm{M}(\mathcal{X}, g) := \langle \mathrm{D}(\mathcal{X}), \cap, \cup, \sim, \varnothing, X \rangle, \quad \text{where}$$

$$\sim U := X \setminus g(U).$$

- For a De Morgan function $f : (\mathcal{X}, g) \to (\mathcal{X}', g')$, put

$$\mathrm{M}(f) = \mathrm{D}(f)$$

Priestley duality for De Morgan algebras [Cornish & Fowler 77]

- A De Morgan space is a pair $(\mathcal{X}, g)$, where
  - $\mathcal{X}$ is a Priestley space;
  - $g : \mathcal{X} \to \mathcal{X}$ is an antimonotonic homeomorphism s.t. $g^2 = id_{\mathcal{X}}$.
- $\mathcal{T}$-morphism $f : (\mathcal{X}, g) \to (\mathcal{X}', g')$ is a De Morgan function if $fg = g'f$.
- For a De Morgan space $(\mathcal{X}, g)$, put:

$$\mathrm{M}(\mathcal{X}, g) := \langle \mathrm{D}(\mathcal{X}), \cap, \cup, \sim, \varnothing, X \rangle, \quad \text{where}$$

$$\sim U := X \setminus g(U).$$

- For a De Morgan function $f : (\mathcal{X}, g) \to (\mathcal{X}', g')$, put

$$\mathrm{M}(f) = \mathrm{D}(f)$$

Priestley duality for De Morgan algebras [Cornish & Fowler 77]

- A De Morgan space is a pair $(\mathcal{X}, g)$, where
  - $\mathcal{X}$ is a Priestley space;
  - $g : \mathcal{X} \to \mathcal{X}$ is an antimonotonic homeomorphism s.t. $g^2 = id_{\mathcal{X}}$.
- $\mathcal{T}$-morphism $f : (\mathcal{X}, g) \to (\mathcal{X}', g')$ is a De Morgan function if $fg = g'f$.
- For a De Morgan space $(\mathcal{X}, g)$, put:

$$\mathrm{M}(\mathcal{X}, g) := \langle \mathrm{D}(\mathcal{X}), \cap, \cup, \sim, \varnothing, X \rangle, \quad \text{where}$$

$$\sim U := X \setminus g(U).$$

- For a De Morgan function $f : (\mathcal{X}, g) \to (\mathcal{X}', g')$, put

$$\mathrm{M}(f) = \mathrm{D}(f)$$

Priestley duality for De Morgan algebras [Cornish & Fowler 77]

- A De Morgan space is a pair $(\mathcal{X}, g)$, where
  - $\mathcal{X}$ is a Priestley space;
  - $g : \mathcal{X} \to \mathcal{X}$ is an antimonotonic homeomorphism s.t. $g^2 = id_{\mathcal{X}}$.
- $\mathcal{T}$-morphism $f : (\mathcal{X}, g) \to (\mathcal{X}', g')$ is a De Morgan function if $fg = g'f$.
- For a De Morgan space $(\mathcal{X}, g)$, put:

$$\mathrm{M}(\mathcal{X}, g) := \langle \mathrm{D}(\mathcal{X}), \cap, \cup, \sim, \varnothing, X \rangle, \quad \text{where}$$

$$\sim U := X \setminus g(U).$$

- For a De Morgan function $f : (\mathcal{X}, g) \to (\mathcal{X}', g')$, put

$$\mathrm{M}(f) = \mathrm{D}(f)$$

Priestley duality for De Morgan algebras [Cornish & Fowler 77]

- For a De Morgan algebra $\mathcal{A}$, put $\mathsf{S}(\mathcal{A}) := (\mathsf{X}(\mathcal{A}), g)$, where

$$g(P) := \mathcal{A} \setminus \widetilde{P}, \quad \widetilde{P} := \{\sim a \mid a \in P\}.$$

- For a homomorphism $f : \mathcal{A} \to \mathcal{B}$ of De Morgan algebras, put

$$\mathsf{S}(f) := \mathsf{X}(f)$$

- Categories $\mathcal{M}^*$ of De Morgan spaces and functions and $\mathcal{M}$ of De Morgan algebras and their homomorphisms are dually equivalent via M and S.

Priestley duality for De Morgan algebras [Cornish & Fowler 77]

- For a De Morgan algebra $\mathcal{A}$, put $\mathsf{S}(\mathcal{A}) := (\mathsf{X}(\mathcal{A}), g)$, where

$$g(P) := \mathcal{A} \setminus \widetilde{P}, \quad \widetilde{P} := \{\sim a \mid a \in P\}.$$

- For a homomorphism $f : \mathcal{A} \to \mathcal{B}$ of De Morgan algebras, put

$$\mathsf{S}(f) := \mathsf{X}(f)$$

- Categories $\mathcal{M}^*$ of De Morgan spaces and functions and $\mathcal{M}$ of De Morgan algebras and their homomorphisms are dually equivalent via M and S.

Priestley duality for De Morgan algebras [Cornish & Fowler 77]

- For a De Morgan algebra $\mathcal{A}$, put $\mathsf{S}(\mathcal{A}) := (\mathsf{X}(\mathcal{A}), g)$, where

$$g(P) := \mathcal{A} \setminus \widetilde{P}, \quad \widetilde{P} := \{\sim a \mid a \in P\}.$$

- For a homomorphism $f : \mathcal{A} \to \mathcal{B}$ of De Morgan algebras, put

$$\mathsf{S}(f) := \mathsf{X}(f)$$

- Categories $\mathcal{M}^*$ of De Morgan spaces and functions and $\mathcal{M}$ of De Morgan algebras and their homomorphisms are dually equivalent via M and S.

Priestley duality for Kleene algebras [Cornish & Fowler 79]

- A De Morgan algebra $\mathcal{A}$ is a Kleene algebra if it satisfies the identity

$$x \wedge \sim x \leq y \vee \sim y$$

- $\mathcal{K}$ is a full subcategory of $\mathcal{M}$, whose objects are Kleene algebras
- A De Morgan space $(\mathcal{X}, g)$ is a Kleene space if

$$X = X^{+} \cup X^{-},$$

where $X^{+} := \{x \mid x \leq g(x)\}$ and $X^{-} := \{x \mid x \geq g(x)\}$
- $\mathcal{K}^{*}$ is a full subcategory of $\mathcal{M}^{*}$, whose objects are Kleene spaces
- Categories $\mathcal{K}^{*}$ and $\mathcal{K}$ are dually equivalent via restrictions of M and S.

Priestley duality for Kleene algebras [Cornish & Fowler 79]

- A De Morgan algebra $\mathcal{A}$ is a Kleene algebra if it satisfies the identity

$$x \wedge \sim x \leq y \vee \sim y$$

- $\mathcal{K}$ is a full subcategory of $\mathcal{M}$, whose objects are Kleene algebras
- A De Morgan space $(\mathcal{X}, g)$ is a Kleene space if

$$X = X^{+} \cup X^{-},$$

where $X^{+} := \{x \mid x \leq g(x)\}$ and $X^{-} := \{x \mid x \geq g(x)\}$

- $\mathcal{K}^{*}$ is a full subcategory of $\mathcal{M}^{*}$, whose objects are Kleene spaces
- Categories $\mathcal{K}^{*}$ and $\mathcal{K}$ are dually equivalent via restrictions of M and S.

Priestley duality for Kleene algebras [Cornish & Fowler 79]

- A De Morgan algebra $\mathcal{A}$ is a Kleene algebra if it satisfies the identity

$$x \wedge \sim x \leq y \vee \sim y$$

- $\mathcal{K}$ is a full subcategory of $\mathcal{M}$, whose objects are Kleene algebras
- A De Morgan space $(\mathcal{X}, g)$ is a Kleene space if

$$X = X^{+} \cup X^{-},$$

where $X^{+} := \{x \mid x \leq g(x)\}$ and $X^{-} := \{x \mid x \geq g(x)\}$

- $\mathcal{K}^{*}$ is a full subcategory of $\mathcal{M}^{*}$, whose objects are Kleene spaces
- Categories $\mathcal{K}^{*}$ and $\mathcal{K}$ are dually equivalent via restrictions of M and S.

Priestley duality for Kleene algebras [Cornish & Fowler 79]

- A De Morgan algebra $\mathcal{A}$ is a Kleene algebra if it satisfies the identity

$$x \wedge \sim x \leq y \vee \sim y$$

- $\mathcal{K}$ is a full subcategory of $\mathcal{M}$, whose objects are Kleene algebras
- A De Morgan space $(\mathcal{X}, g)$ is a Kleene space if

$$X = X^{+} \cup X^{-},$$

where $X^{+} := \{x \mid x \leq g(x)\}$ and $X^{-} := \{x \mid x \geq g(x)\}$

- $\mathcal{K}^{*}$ is a full subcategory of $\mathcal{M}^{*}$, whose objects are Kleene spaces
- Categories $\mathcal{K}^{*}$ and $\mathcal{K}$ are dually equivalent via restrictions of M and S.

Priestley duality for Kleene algebras [Cornish & Fowler 79]

- A De Morgan algebra $\mathcal{A}$ is a Kleene algebra if it satisfies the identity

$$x \wedge \sim x \leq y \vee \sim y$$

- $\mathcal{K}$ is a full subcategory of $\mathcal{M}$, whose objects are Kleene algebras
- A De Morgan space $(\mathcal{X}, g)$ is a Kleene space if

$$X = X^+ \cup X^-,$$

where $X^+ := \{x \mid x \leq g(x)\}$ and $X^- := \{x \mid x \geq g(x)\}$
- $\mathcal{K}^*$ is a full subcategory of $\mathcal{M}^*$, whose objects are Kleene spaces
- Categories $\mathcal{K}^*$ and $\mathcal{K}$ are dually equivalent via restrictions of M and S.

bounded **N4**-lattices

$\mathcal{A} = \langle A, \vee, \wedge, \to, \sim, \bot, 1\rangle$ is a bounded **N4**-lattice if:

1. $\langle A, \vee, \wedge, \sim, \bot, 1\rangle$ is a De Morgan algebra;
2. $\preceq$, where $a \preceq b$ denotes $(a \to b) \to (a \to b) = a \to b$, is a preordering on $\mathcal{A}$;
3. $\approx := \preceq \cap \preceq^{-1}$ is a congruence wrt $\vee, \wedge, \to$ and $\mathcal{A}_{\bowtie} := \langle A, \vee, \wedge, \to, \bot, 1\rangle / \approx$ is a Heyting algebra;
4. $\sim (a \to b) \approx a \wedge \sim b$;
5. $a \leq b$ if and only if $a \preceq b$ and $\sim b \preceq \sim a$.

bounded **N4**-lattices

$\mathcal{A} = \langle A, \vee, \wedge, \to, \sim, \bot, 1\rangle$ is a bounded **N4**-lattice if:

1. $\langle A, \vee, \wedge, \sim, \bot, 1\rangle$ is a De Morgan algebra;
2. $\preceq$, where $a \preceq b$ denotes $(a \to b) \to (a \to b) = a \to b$, is a preordering on $\mathcal{A}$;
3. $\approx := \preceq \cap \preceq^{-1}$ is a congruence wrt $\vee, \wedge, \to$ and $\mathcal{A}_{\bowtie} := \langle A, \vee, \wedge, \to, \bot, 1\rangle / \approx$ is a Heyting algebra;
4. $\sim (a \to b) \approx a \wedge \sim b$;
5. $a \leq b$ if and only if $a \preceq b$ and $\sim b \preceq \sim a$.

N3-lattices [Rasiowa58]

1. An **N3**-lattice $\mathcal{A}$ is a bounded **N4**-lattice s.t. $a \wedge \sim a \preceq b$ for $a, b \in \mathcal{A}$
2. in which case $a \preceq b \Leftrightarrow a \to b = 1$
3. $\mathcal{A} \models \varphi$ iff $\varphi = 1$ is an identity on $\mathcal{A}$
4. **N3**-lattices are Kleene algebras
5. weak implication via relative pseudo complement in **N3**-lattices

   $a \to b = a \supset (\sim a \vee b)$, where $a \supset b := \sup\{x \mid a \wedge x \leq b\}$
6. similar formula for bounded **N4**-lattices

   $a \to b = \sim \neg a \supset (\neg a \vee b)$, where $\neg a = a \to \bot$

N3-lattices [Rasiowa58]

1. An **N3**-lattice $\mathcal{A}$ is a bounded **N4**-lattice s.t. $a \wedge \sim a \preceq b$ for $a, b \in \mathcal{A}$
2. in which case $a \preceq b \Leftrightarrow a \to b = 1$
3. $\mathcal{A} \models \varphi$ iff $\varphi = 1$ is an identity on $\mathcal{A}$
4. **N3**-lattices are Kleene algebras
5. weak implication via relative pseudo complement in **N3**-lattices

   $a \to b = a \supset (\sim a \vee b)$, where $a \supset b := \sup\{x \mid a \wedge x \leq b\}$
6. similar formula for bounded **N4**-lattices

   $a \to b = \sim \neg a \supset (\neg a \vee b)$, where $\neg a = a \to \perp$

N3-lattices [Rasiowa58]

1. An **N3**-lattice $\mathcal{A}$ is a bounded **N4**-lattice s.t. $a \wedge \sim a \preceq b$ for $a, b \in \mathcal{A}$
2. in which case $a \preceq b \Leftrightarrow a \to b = 1$
3. $\mathcal{A} \models \varphi$ iff $\varphi = 1$ is an identity on $\mathcal{A}$
4. **N3**-lattices are Kleene algebras
5. weak implication via relative pseudo complement in **N3**-lattices

   $a \to b = a \supset (\sim a \vee b)$, where $a \supset b := \sup\{x \mid a \wedge x \leq b\}$

6. similar formula for bounded **N4**-lattices

   $a \to b = \sim \neg a \supset (\neg a \vee b)$, where $\neg a = a \to \bot$

N3-lattices [Rasiowa58]

1. An **N3**-lattice $\mathcal{A}$ is a bounded **N4**-lattice s.t. $a \wedge \sim a \preceq b$ for $a, b \in \mathcal{A}$
2. in which case $a \preceq b \Leftrightarrow a \rightarrow b = 1$
3. $\mathcal{A} \models \varphi$ iff $\varphi = 1$ is an identity on $\mathcal{A}$
4. **N3**-lattices are Kleene algebras
5. weak implication via relative pseudo complement in **N3**-lattices

   $a \rightarrow b = a \supset (\sim a \vee b)$, where $a \supset b := \sup\{x \mid a \wedge x \leq b\}$

6. similar formula for bounded **N4**-lattices

   $a \rightarrow b = \sim \neg a \supset (\neg a \vee b)$, where $\neg a = a \rightarrow \bot$

N3-lattices [Rasiowa58]

1. An N3-lattice $\mathcal{A}$ is a bounded **N4**-lattice s.t. $a \wedge \sim a \preceq b$ for $a, b \in \mathcal{A}$
2. in which case $a \preceq b \Leftrightarrow a \to b = 1$
3. $\mathcal{A} \models \varphi$ iff $\varphi = 1$ is an identity on $\mathcal{A}$
4. **N3**-lattices are Kleene algebras
5. weak implication via relative pseudo complement in **N3**-lattices

   $a \to b = a \supset (\sim a \vee b)$, where $a \supset b := \sup\{x \mid a \wedge x \leq b\}$
6. similar formula for bounded **N4**-lattices

   $a \to b = \sim \neg a \supset (\neg a \vee b)$, where $\neg a = a \to \bot$

N3-lattices [Rasiowa58]

1. An **N3**-lattice $\mathcal{A}$ is a bounded **N4**-lattice s.t. $a \wedge \sim a \preceq b$ for $a, b \in \mathcal{A}$
2. in which case $a \preceq b \Leftrightarrow a \to b = 1$
3. $\mathcal{A} \models \varphi$ iff $\varphi = 1$ is an identity on $\mathcal{A}$
4. **N3**-lattices are Kleene algebras
5. weak implication via relative pseudo complement in **N3**-lattices

   $a \to b = a \supset (\sim a \vee b)$, where $a \supset b := \sup\{x \mid a \wedge x \leq b\}$
6. similar formula for bounded **N4**-lattices

   $a \to b = \sim \neg a \supset (\neg a \vee b)$, where $\neg a = a \to \bot$

Special filters [Rasiowa58] on **N4**-lattices

- Let $\mathcal{A}$ be an **N4**-lattice and $\varnothing \neq F \subseteq A$.
- F is a special filter of the first kind (*sffk*) on $\mathcal{A}$ if:
  1. $a \wedge b \in F$ for any $a, b \in F$,
  2. $a \in F$ and $a \preceq b$ implies $b \in F$.
- The lattice of *sffk* is isomorphic to the lattice of filters of $\mathcal{A}_{\bowtie}$
- F is a special filter of the second kind (*sfsk*) on $\mathcal{A}$ if:
  1. $a \wedge b \in F$ for any $a, b \in F$,
  2. $a \in F$ and $\sim b \preceq \sim a$ implies $b \in F$.
- The lattice of *sfsk* is isomorphic to the lattice of ideals of $\mathcal{A}_{\bowtie}$
- Every prime (lattice) filter on $\mathcal{A}$ is either *sffk* or *sfsk*.

Special filters [Rasiowa58] on **N4**-lattices

- Let $\mathcal{A}$ be an **N4**-lattice and $\varnothing \neq F \subseteq A$.
- F is a special filter of the first kind (*sffk*) on $\mathcal{A}$ if:
  1. $a \wedge b \in F$ for any $a, b \in F$,
  2. $a \in F$ and $a \preceq b$ implies $b \in F$.
- The lattice of *sffk* is isomorphic to the lattice of filters of $\mathcal{A}_{\bowtie}$
- F is a special filter of the second kind (*sfsk*) on $\mathcal{A}$ if:
  1. $a \wedge b \in F$ for any $a, b \in F$,
  2. $a \in F$ and $\sim b \preceq \sim a$ implies $b \in F$.
- The lattice of *sfsk* is isomorphic to the lattice of ideals of $\mathcal{A}_{\bowtie}$
- Every prime (lattice) filter on $\mathcal{A}$ is either *sffk* or *sfsk*.

Special filters [Rasiowa58] on **N4**-lattices

- Let $\mathcal{A}$ be an **N4**-lattice and $\varnothing \neq F \subseteq A$.
- F is a special filter of the first kind (*sffk*) on $\mathcal{A}$ if:
  1. $a \wedge b \in F$ for any $a, b \in F$,
  2. $a \in F$ and $a \preceq b$ implies $b \in F$.
- The lattice of *sffk* is isomorphic to the lattice of filters of $\mathcal{A}_{\bowtie}$
- F is a special filter of the second kind (*sfsk*) on $\mathcal{A}$ if:
  1. $a \wedge b \in F$ for any $a, b \in F$,
  2. $a \in F$ and $\sim b \preceq \sim a$ implies $b \in F$.
- The lattice of *sfsk* is isomorphic to the lattice of ideals of $\mathcal{A}_{\bowtie}$
- Every prime (lattice) filter on $\mathcal{A}$ is either *sffk* or *sfsk*.

Special filters [Rasiowa58] on **N4**-lattices

- Let $\mathcal{A}$ be an **N4**-lattice and $\varnothing \neq F \subseteq A$.
- F is a special filter of the first kind (*sffk*) on $\mathcal{A}$ if:
  1. $a \wedge b \in F$ for any $a, b \in F$,
  2. $a \in F$ and $a \preceq b$ implies $b \in F$.
- The lattice of *sffk* is isomorphic to the lattice of filters of $\mathcal{A}_{\bowtie}$
- F is a special filter of the second kind (*sfsk*) on $\mathcal{A}$ if:
  1. $a \wedge b \in F$ for any $a, b \in F$,
  2. $a \in F$ and $\sim b \preceq \sim a$ implies $b \in F$.
- The lattice of *sfsk* is isomorphic to the lattice of ideals of $\mathcal{A}_{\bowtie}$
- Every prime (lattice) filter on $\mathcal{A}$ is either *sffk* or *sfsk*.

Special filters [Rasiowa58] on **N4**-lattices

- Let $\mathcal{A}$ be an **N4**-lattice and $\varnothing \neq F \subseteq A$.
- F is a special filter of the first kind (*sffk*) on $\mathcal{A}$ if:
  1. $a \wedge b \in F$ for any $a, b \in F$,
  2. $a \in F$ and $a \preceq b$ implies $b \in F$.
- The lattice of *sffk* is isomorphic to the lattice of filters of $\mathcal{A}_{\bowtie}$
- F is a special filter of the second kind (*sfsk*) on $\mathcal{A}$ if:
  1. $a \wedge b \in F$ for any $a, b \in F$,
  2. $a \in F$ and $\sim b \preceq \sim a$ implies $b \in F$.
- The lattice of *sfsk* is isomorphic to the lattice of ideals of $\mathcal{A}_{\bowtie}$
- Every prime (lattice) filter on $\mathcal{A}$ is either *sffk* or *sfsk*.

Special filters [Rasiowa58] on **N4**-lattices

- Let $\mathcal{A}$ be an **N4**-lattice and $\varnothing \neq F \subseteq A$.
- F is a special filter of the first kind (*sffk*) on $\mathcal{A}$ if:
  1. $a \wedge b \in F$ for any $a, b \in F$,
  2. $a \in F$ and $a \preceq b$ implies $b \in F$.
- The lattice of *sffk* is isomorphic to the lattice of filters of $\mathcal{A}_{\bowtie}$
- F is a special filter of the second kind (*sfsk*) on $\mathcal{A}$ if:
  1. $a \wedge b \in F$ for any $a, b \in F$,
  2. $a \in F$ and $\sim b \preceq \sim a$ implies $b \in F$.
- The lattice of *sfsk* is isomorphic to the lattice of ideals of $\mathcal{A}_{\bowtie}$
- Every prime (lattice) filter on $\mathcal{A}$ is either *sffk* or *sfsk*.

De Morgan space of a bounded **N4**-lattice $\mathcal{A}$

1. Since $\mathcal{A}$ is a De Morgan algebra, $(\mathrm{X}(\mathcal{A}), g)$, where $g(P) = \mathcal{A} \setminus \tilde{P}$, $\tilde{P} = \{\sim a \mid a \in P\}$, is a De Morgan space.
2. Put
$$\begin{aligned}
\mathrm{X}^1(\mathcal{A}) &= \{P \in \mathrm{X}(\mathcal{A}) \mid P \text{ is a } \mathit{sffk}\} \\
\mathrm{X}^2(\mathcal{A}) &= \{P \in \mathrm{X}(\mathcal{A}) \mid P \text{ is a } \mathit{sfsk}\} \\
\mathrm{X}^+(\mathcal{A}) &= \{P \in \mathrm{X}(\mathcal{A}) \mid P \subseteq g(P)\} \\
\mathrm{X}^-(\mathcal{A}) &= \{P \in \mathrm{X}(\mathcal{A}) \mid g(P) \subseteq P\}
\end{aligned}$$
3. $g(\mathrm{X}^1(\mathcal{A})) = \mathrm{X}^2(\mathcal{A})$ and $g(\mathrm{X}^+(\mathcal{A})) = \mathrm{X}^-(\mathcal{A})$
4. $\mathcal{A}$ is an **N3**-lattice iff
$$\mathrm{X}^1(\mathcal{A}) = \mathrm{X}^+(\mathcal{A}) \text{ and } \mathrm{X}^2(\mathcal{A}) = \mathrm{X}^-(\mathcal{A})$$

De Morgan space of a bounded **N4**-lattice $\mathcal{A}$

1. Since $\mathcal{A}$ is a De Morgan algebra, $(\mathsf{X}(\mathcal{A}), g)$, where $g(P) = \mathcal{A} \setminus \tilde{P}$, $\tilde{P} = \{\sim a \mid a \in P\}$, is a De Morgan space.
2. Put

$$
\begin{aligned}
\mathsf{X}^1(\mathcal{A}) &= \{P \in \mathsf{X}(\mathcal{A}) \mid P \text{ is a } \textit{sffk}\} \\
\mathsf{X}^2(\mathcal{A}) &= \{P \in \mathsf{X}(\mathcal{A}) \mid P \text{ is a } \textit{sfsk}\} \\
\mathsf{X}^+(\mathcal{A}) &= \{P \in \mathsf{X}(\mathcal{A}) \mid P \subseteq g(P)\} \\
\mathsf{X}^-(\mathcal{A}) &= \{P \in \mathsf{X}(\mathcal{A}) \mid g(P) \subseteq P\}
\end{aligned}
$$

3. $g(\mathsf{X}^1(\mathcal{A})) = \mathsf{X}^2(\mathcal{A})$ and $g(\mathsf{X}^+(\mathcal{A})) = \mathsf{X}^-(\mathcal{A})$
4. $\mathcal{A}$ is an **N3**-lattice iff

$$\mathsf{X}^1(\mathcal{A}) = \mathsf{X}^+(\mathcal{A}) \text{ and } \mathsf{X}^2(\mathcal{A}) = \mathsf{X}^-(\mathcal{A})$$

De Morgan space of a bounded **N4**-lattice $\mathcal{A}$

1. Since $\mathcal{A}$ is a De Morgan algebra, $(\mathrm{X}(\mathcal{A}), g)$, where $g(P) = \mathcal{A} \setminus \tilde{P}$, $\tilde{P} = \{\sim a \mid a \in P\}$, is a De Morgan space.
2. Put

$$\begin{aligned}
\mathrm{X}^1(\mathcal{A}) &= \{P \in \mathrm{X}(\mathcal{A}) \mid P \text{ is a } sffk\} \\
\mathrm{X}^2(\mathcal{A}) &= \{P \in \mathrm{X}(\mathcal{A}) \mid P \text{ is a } sfsk\} \\
\mathrm{X}^+(\mathcal{A}) &= \{P \in \mathrm{X}(\mathcal{A}) \mid P \subseteq g(P)\} \\
\mathrm{X}^-(\mathcal{A}) &= \{P \in \mathrm{X}(\mathcal{A}) \mid g(P) \subseteq P\}
\end{aligned}$$

3. $g(\mathrm{X}^1(\mathcal{A})) = \mathrm{X}^2(\mathcal{A})$ and $g(\mathrm{X}^+(\mathcal{A})) = \mathrm{X}^-(\mathcal{A})$
4. $\mathcal{A}$ is an **N3**-lattice iff

$$\mathrm{X}^1(\mathcal{A}) = \mathrm{X}^+(\mathcal{A}) \text{ and } \mathrm{X}^2(\mathcal{A}) = \mathrm{X}^-(\mathcal{A})$$

De Morgan space of a bounded **N4**-lattice $\mathcal{A}$

1. Since $\mathcal{A}$ is a De Morgan algebra, $(X(\mathcal{A}), g)$, where $g(P) = \mathcal{A} \setminus \tilde{P}$, $\tilde{P} = \{\sim a \mid a \in P\}$, is a De Morgan space.
2. Put

$$\begin{aligned} X^1(\mathcal{A}) &= \{P \in X(\mathcal{A}) \mid P \text{ is a } \textit{sffk}\} \\ X^2(\mathcal{A}) &= \{P \in X(\mathcal{A}) \mid P \text{ is a } \textit{sfsk}\} \\ X^+(\mathcal{A}) &= \{P \in X(\mathcal{A}) \mid P \subseteq g(P)\} \\ X^-(\mathcal{A}) &= \{P \in X(\mathcal{A}) \mid g(P) \subseteq P\} \end{aligned}$$

3. $g(X^1(\mathcal{A})) = X^2(\mathcal{A})$ and $g(X^+(\mathcal{A})) = X^-(\mathcal{A})$
4. $\mathcal{A}$ is an **N3**-lattice iff
$$X^1(\mathcal{A}) = X^+(\mathcal{A}) \text{ and } X^2(\mathcal{A}) = X^-(\mathcal{A})$$

$N3$-spaces [Cignoli 86, Sendlewski 90]

$\mathcal{X} = (X, \leq, \tau, g)$, where X is a set, $\leq$ a partial order on X, $g : X \to X$, and τ is a topology on X, is an $N3$-space if:

1. $(X, \leq, \tau, g)$ is a Kleene space, $X = X^+ \cup X^-$, where $X^+ := \{x \mid x \leq g(x)\}$ and $X^- := \{x \mid x \geq g(x)\}$;
2. $(X^+, \leq\restriction_{X^+}, \tau\restriction_{X^+})$ is a Heyting space;
3. (Monteiro Interpolation Property) for any $x \in X^+$ and $y \in X^-$, if $x \leq y$, then there exists $z \in X$ s.t. $x, g(y) \leq z \leq g(x), y$

$N3$-spaces [Cignoli 86, Sendlewski 90]

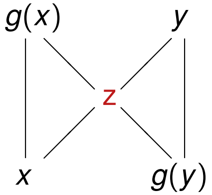
$\mathcal{X} = (X, \leq, \tau, g)$, where X is a set, $\leq$ a partial order on X, $g : X \to X$, and τ is a topology on X, is an $N3$-space if:

1. $(X, \leq, \tau, g)$ is a Kleene space, $X = X^+ \cup X^-$, where $X^+ := \{x \mid x \leq g(x)\}$ and $X^- := \{x \mid x \geq g(x)\}$;
2. $(X^+, \leq\restriction_{X^+}, \tau\restriction_{X^+})$ is a Heyting space;
3. (Monteiro Interpolation Property) for any $x \in X^+$ and $y \in X^-$, if $x \leq y$, then there exists $z \in X$ s.t. $x, g(y) \leq z \leq g(x), y$

$N3$-spaces [Cignoli 86, Sendlewski 90]

$\mathcal{X} = (X, \leq, \tau, g)$, where X is a set, $\leq$ a partial order on X, $g : X \to X$, and τ is a topology on X, is an $N3$-space if:

1. $(X, \leq, \tau, g)$ is a Kleene space, $X = X^+ \cup X^-$, where $X^+ := \{x \mid x \leq g(x)\}$ and $X^- := \{x \mid x \geq g(x)\}$;
2. $(X^+, \leq\restriction_{X^+}, \tau \restriction_{X^+})$ is a Heyting space;
3. (Monteiro Interpolation Property) for any $x \in X^+$ and $y \in X^-$, if $x \leq y$, then there exists $z \in X$ s.t. $x, g(y) \leq z \leq g(x), y$



$N3$-function [Cignoli 86, Sendlewski 90]

$f : (X, \leq, \tau, g) \to (Y, \leq', \tau', g')$ is an $N3$-function if:

1. $f : (X \leq, \tau, g) \to (Y, \leq', \tau', g')$ is a De Morgan function, i.e., an order preserving continuous mapping s. t. $fg = g'f$;
2. $f \restriction_{X^+}$ is a Heyting function from X^+ to Y^+.

N3-Duality [Cignoli 86, Sendlewski 90]

1. Let $\mathcal{A}$ be an **N3**-lattice. Then De Morgan space

$$\mathsf{M}(\mathcal{A}) := (\mathsf{X}(\mathcal{A}), \subseteq, \tau_{\mathcal{A}}, g_{\mathcal{A}})$$

is an *N*3-space

2. If $f : \mathcal{A} \to \mathcal{B}$ is a homomorphism of **N3**-lattices, then

$$\mathsf{M}(f) : \mathsf{M}(\mathcal{B}) \to \mathsf{M}(\mathcal{A})$$

is an *N*3-function.

N3-Duality [Cignoli 86, Sendlewski 90]

1. Let $\mathcal{A}$ be an **N3**-lattice. Then De Morgan space

$$\mathrm{M}(\mathcal{A}) := (\mathrm{X}(\mathcal{A}), \subseteq, \tau_{\mathcal{A}}, g_{\mathcal{A}})$$

is an *N*3-space

2. If $f : \mathcal{A} \to \mathcal{B}$ is a homomorphism of **N3**-lattices, then

$$\mathrm{M}(f) : \mathrm{M}(\mathcal{B}) \to \mathrm{M}(\mathcal{A})$$

is an *N*3-function.

N3-Duality [Cignoli 86, Sendlewski 90]

1. Let $\mathcal{X} = (X, \leq, \tau, g)$ be an *N*3-space. Then $\mathrm{N}^3(\mathcal{X}) = \langle \mathrm{D}(\mathcal{X}), \cup, \cap, \to, \sim, \varnothing, X \rangle$, where

$$\sim U := X \setminus g(U),$$

$$U \to V := X \setminus ((U \cap g(U)) \setminus V) \downarrow$$

(compare with $a \to b = a \supset (\sim a \vee b)$)

is an **N3**-lattice.

2. For an *N*3-function $f : (X, \leq, \tau, g) \to (Y, \leq', \tau', g')$, put

$$\mathrm{N}^3(f) := \mathrm{D}(f)$$

3. Categories $\mathcal{N}_3^*$ of *N*3-spaces and *N*3-functions and $\mathcal{N}_3$ of **N3**-lattices and their homomorphisms are dually equivalent via N^3 and the restriction of S.

N3-Duality [Cignoli 86, Sendlewski 90]

1. Let $\mathcal{X} = (X, \leq, \tau, g)$ be an *N*3-space. Then $\mathrm{N}^3(\mathcal{X}) = \langle \mathrm{D}(\mathcal{X}), \cup, \cap, \to, \sim, \varnothing, X \rangle$, where

$$\sim U := X \setminus g(U),$$

$$U \to V := X \setminus ((U \cap g(U)) \setminus V) \downarrow$$

(compare with $a \to b = a \supset (\sim a \vee b)$)

is an **N3**-lattice.

2. For an *N*3-function $f : (X, \leq, \tau, g) \to (Y, \leq', \tau', g')$, put

$$\mathrm{N}^3(f) := \mathrm{D}(f)$$

3. Categories $\mathcal{N}_3^*$ of *N*3-spaces and *N*3-functions and $\mathcal{N}_3$ of **N3**-lattices and their homomorphisms are dually equivalent via N^3 and the restriction of S.

N3-Duality [Cignoli 86, Sendlewski 90]

1. Let $\mathcal{X} = (X, \leq, \tau, g)$ be an *N*3-space. Then $\mathrm{N}^3(\mathcal{X}) = \langle \mathrm{D}(\mathcal{X}), \cup, \cap, \to, \sim, \varnothing, X \rangle$, where

$$\sim U := X \setminus g(U),$$

$$U \to V := X \setminus ((U \cap g(U)) \setminus V) \downarrow$$

$$\text{(compare with } a \to b = a \supset (\sim a \vee b))$$

is an **N3**-lattice.

2. For an *N*3-function $f : (X, \leq, \tau, g) \to (Y, \leq', \tau', g')$, put

$$\mathrm{N}^3(f) := \mathrm{D}(f)$$

3. Categories $\mathcal{N}_3^*$ of *N*3-spaces and *N*3-functions and $\mathcal{N}_3$ of **N3**-lattices and their homomorphisms are dually equivalent via N^3 and the restriction of S.

$N4$-spaces

Consider a tuple $\mathcal{X} = (X, X^1, \leq, \tau, g)$, where X is a set, $X^1 \subseteq X$, $\leq$ a partial order on X, $g : X \to X$, and τ is a topology on X. Put

$$X^2 := g(X^1),\ X^+ := \{x \in X \mid x \leq g(x)\},\ X^- := \{x \in X \mid g(x) \leq x\}.$$

$\mathcal{X}$ is said to be an $N4$-space if:

1. $(X, \leq, \tau, g)$ is a De Morgan space;
2. X^1 is closed in τ, $X = X^1 \cup X^2$, and $X^1 \cap X^2 = X^+ \cap X^-$;
3. $(X^1, \leq\restriction_{X^1}, \tau\restriction_{X^1})$ is a Heyting space;
4. for any $x \in X^1$ and $y \in X^2$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and there exists $z \in X$ s.t. $x, g(y) \leq z \leq g(x), y$;
5. for any $x \in X^2$ and $y \in X^1$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and $x \leq g(y)$.

$N4$-spaces

Consider a tuple $\mathcal{X} = (X, X^1, \leq, \tau, g)$, where X is a set, $X^1 \subseteq X$, $\leq$ a partial order on X, $g : X \to X$, and τ is a topology on X. Put

$$X^2 := g(X^1),\ X^+ := \{x \in X \mid x \leq g(x)\},\ X^- := \{x \in X \mid g(x) \leq x\}.$$

$\mathcal{X}$ is said to be an $N4$-space if:

1. $(X, \leq, \tau, g)$ is a De Morgan space;
2. X^1 is closed in τ, $X = X^1 \cup X^2$, and $X^1 \cap X^2 = X^+ \cap X^-$;
3. $(X^1, \leq\restriction_{X^1}, \tau\restriction_{X^1})$ is a Heyting space;
4. for any $x \in X^1$ and $y \in X^2$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and there exists $z \in X$ s.t. $x, g(y) \leq z \leq g(x), y$;
5. for any $x \in X^2$ and $y \in X^1$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and $x \leq g(y)$.

$N4$-spaces

Consider a tuple $\mathcal{X} = (X, X^1, \leq, \tau, g)$, where X is a set, $X^1 \subseteq X$, $\leq$ a partial order on X, $g : X \to X$, and τ is a topology on X. Put

$$X^2 := g(X^1),\ X^+ := \{x \in X \mid x \leq g(x)\},\ X^- := \{x \in X \mid g(x) \leq x\}.$$

$\mathcal{X}$ is said to be an $N4$-space if:

1. $(X, \leq, \tau, g)$ is a De Morgan space;
2. X^1 is closed in τ, $X = X^1 \cup X^2$, and $X^1 \cap X^2 = X^+ \cap X^-$;
3. $(X^1, \leq\upharpoonright_{X^1}, \tau\upharpoonright_{X^1})$ is a Heyting space;
4. for any $x \in X^1$ and $y \in X^2$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and there exists $z \in X$ s.t. $x, g(y) \leq z \leq g(x), y$;
5. for any $x \in X^2$ and $y \in X^1$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and $x \leq g(y)$.

$N4$-spaces

Consider a tuple $\mathcal{X} = (X, X^1, \leq, \tau, g)$, where X is a set, $X^1 \subseteq X$, $\leq$ a partial order on X, $g : X \to X$, and τ is a topology on X. Put

$$X^2 := g(X^1),\ X^+ := \{x \in X \mid x \leq g(x)\},\ X^- := \{x \in X \mid g(x) \leq x\}.$$

$\mathcal{X}$ is said to be an $N4$-space if:

1. $(X, \leq, \tau, g)$ is a De Morgan space;
2. X^1 is closed in τ, $X = X^1 \cup X^2$, and $X^1 \cap X^2 = X^+ \cap X^-$;
3. $(X^1, \leq\restriction_{X^1}, \tau\restriction_{X^1})$ is a Heyting space;
4. for any $x \in X^1$ and $y \in X^2$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and there exists $z \in X$ s.t. $x, g(y) \leq z \leq g(x), y$;
5. for any $x \in X^2$ and $y \in X^1$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and $x \leq g(y)$.

$N4$-spaces

Consider a tuple $\mathcal{X} = (X, X^1, \leq, \tau, g)$, where X is a set, $X^1 \subseteq X$, $\leq$ a partial order on X, $g : X \to X$, and τ is a topology on X. Put

$$X^2 := g(X^1),\ X^+ := \{x \in X \mid x \leq g(x)\},\ X^- := \{x \in X \mid g(x) \leq x\}.$$

$\mathcal{X}$ is said to be an $N4$-space if:

1. $(X, \leq, \tau, g)$ is a De Morgan space;
2. X^1 is closed in τ, $X = X^1 \cup X^2$, and $X^1 \cap X^2 = X^+ \cap X^-$;
3. $(X^1, \leq\restriction_{X^1}, \tau\restriction_{X^1})$ is a Heyting space;
4. for any $x \in X^1$ and $y \in X^2$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and there exists $z \in X$ s.t. $x, g(y) \leq z \leq g(x), y$;
5. for any $x \in X^2$ and $y \in X^1$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and $x \leq g(y)$.

$N4$-spaces

Consider a tuple $\mathcal{X} = (X, X^1, \leq, \tau, g)$, where X is a set, $X^1 \subseteq X$, $\leq$ a partial order on X, $g : X \to X$, and τ is a topology on X. Put

$$X^2 := g(X^1),\ X^+ := \{x \in X \mid x \leq g(x)\},\ X^- := \{x \in X \mid g(x) \leq x\}.$$

$\mathcal{X}$ is said to be an $N4$-space if:

1. $(X, \leq, \tau, g)$ is a De Morgan space;
2. X^1 is closed in τ, $X = X^1 \cup X^2$, and $X^1 \cap X^2 = X^+ \cap X^-$;
3. $(X^1, \leq\restriction_{X^1}, \tau\restriction_{X^1})$ is a Heyting space;
4. for any $x \in X^1$ and $y \in X^2$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and there exists $z \in X$ s.t. $x, g(y) \leq z \leq g(x), y$;
5. for any $x \in X^2$ and $y \in X^1$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and $x \leq g(y)$.

$N4$-spaces

Consider a tuple $\mathcal{X} = (X, X^1, \leq, \tau, g)$, where X is a set, $X^1 \subseteq X$, $\leq$ a partial order on X, $g : X \to X$, and τ is a topology on X. Put

$$X^2 := g(X^1),\ X^+ := \{x \in X \mid x \leq g(x)\},\ X^- := \{x \in X \mid g(x) \leq x\}.$$

$\mathcal{X}$ is said to be an $N4$-space if:

1. $(X, \leq, \tau, g)$ is a De Morgan space;
2. X^1 is closed in τ, $X = X^1 \cup X^2$, and $X^1 \cap X^2 = X^+ \cap X^-$;
3. $(X^1, \leq\restriction_{X^1}, \tau\restriction_{X^1})$ is a Heyting space;
4. for any $x \in X^1$ and $y \in X^2$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and there exists $z \in X$ s.t. $x, g(y) \leq z \leq g(x), y$;
5. for any $x \in X^2$ and $y \in X^1$, if $x \leq y$, then $x \in X^+$, $y \in X^-$, and $x \leq g(y)$.

$N4$-function

$f : (X, X^1, \leq, \tau, g) \to (Y, Y^1, \leq', \tau', g')$ is an $N4$-function if:

1. $f : (X \leq, \tau, g) \to (Y, \leq', \tau', g')$ is a De Morgan function, i.e., an order preserving continuous mapping s. t. $fg = g'f$;
2. $f(X^1) \subseteq Y^1$;
3. $f \restriction_{X^1}$ is a Heyting function.

Duality

1. Let $\mathcal{A}$ be a bounded **N4**-lattice. Then

$$\mathsf{O}(\mathcal{A}) := (\mathsf{X}(\mathcal{A}), \mathsf{X}^1(\mathcal{A}), \subseteq, \tau_\mathcal{A}, g_\mathcal{A}),$$

where

- $\mathsf{X}(\mathcal{A})$ are all prime filters on $\mathcal{A}$
- $\mathsf{X}^1(\mathcal{A})$ are all *sffk* in $\mathsf{X}(\mathcal{A})$
- $\tau_\mathcal{A}$ and $g_\mathcal{A}$ as for De Morgan algebras

is an *N4-space*

2. If $f : \mathcal{A} \to \mathcal{B}$ is a homomorphism of bounded **N4**-lattices, then

$$\mathsf{O}(f) := \mathsf{X}(f)$$

is an *N4-function*.

Duality

1. Let $\mathcal{A}$ be a bounded **N4**-lattice. Then

$$\mathsf{O}(\mathcal{A}) := (\mathsf{X}(\mathcal{A}), \mathsf{X}^1(\mathcal{A}), \subseteq, \tau_{\mathcal{A}}, g_{\mathcal{A}}),$$

   where
   - $\mathsf{X}(\mathcal{A})$ are all prime filters on $\mathcal{A}$
   - $\mathsf{X}^1(\mathcal{A})$ are all *sffk* in $\mathsf{X}(\mathcal{A})$
   - $\tau_{\mathcal{A}}$ and $g_{\mathcal{A}}$ as for De Morgan algebras

   is an *N4-space*

2. If $f : \mathcal{A} \to \mathcal{B}$ is a homomorphism of bounded **N4**-lattices, then

$$\mathsf{O}(f) := \mathsf{X}(f)$$

   is an *N4-function*.

Duality

1. Let $\mathcal{X} = (X, X^1, \leq, \tau, g)$ be an $N4$-space. Then $\mathrm{N}(\mathcal{X}) = \langle \mathrm{D}(\mathcal{X}), \cup, \cap, \to, \sim, \varnothing, X \rangle$, where

$$\sim U := X \setminus g(U),$$

$$U \to V := (X^1 \setminus ((U \setminus V) \cap X^1) \downarrow) \cup (X^2 \setminus (g(U) \setminus V)),$$

$$\text{(compare with } \ (a, b) \to (c, d) = (a \to c, a \wedge d))$$

is an $\mathbf{N4}^{\perp}$-lattice.

2. For an $N4$-function $f : (X, X^1, \leq, \tau, g) \to (Y, Y^1, \leq', \tau', g')$, put

$$\mathrm{N}(f) := \mathrm{D}(f)$$

3. Categories $\mathcal{N}_4^*$ of $N4$-spaces and $N4$-functions and $\mathcal{N}_4$ of bounded **N4**-lattices and their homomorphisms are dually equivalent via N and the restriction of O.

Duality

1. Let $\mathcal{X} = (X, X^1, \leq, \tau, g)$ be an $N4$-space. Then $\mathrm{N}(\mathcal{X}) = \langle \mathrm{D}(\mathcal{X}), \cup, \cap, \to, \sim, \varnothing, X \rangle$, where

$$\sim U := X \setminus g(U),$$

$$U \to V := (X^1 \setminus ((U \setminus V) \cap X^1) \downarrow) \cup (X^2 \setminus (g(U) \setminus V)),$$

$$\text{(compare with } (a, b) \to (c, d) = (a \to c, a \wedge d))$$

is an $\mathbf{N4}^{\perp}$-lattice.

2. For an $N4$-function $f : (X, X^1, \leq, \tau, g) \to (Y, Y^1, \leq', \tau', g')$, put

$$\mathrm{N}(f) := \mathrm{D}(f)$$

3. Categories $\mathcal{N}_4^*$ of $N4$-spaces and $N4$-functions and $\mathcal{N}_4$ of bounded $\mathbf{N4}$-lattices and their homomorphisms are dually equivalent via N and the restriction of O.

Duality

1. Let $\mathcal{X} = (X, X^1, \leq, \tau, g)$ be an $N4$-space. Then $\mathsf{N}(\mathcal{X}) = \langle \mathsf{D}(\mathcal{X}), \cup, \cap, \to, \sim, \varnothing, X \rangle$, where

$$\sim U := X \setminus g(U),$$

$$U \to V := (X^1 \setminus ((U \setminus V) \cap X^1) \downarrow) \cup (X^2 \setminus (g(U) \setminus V)),$$

$$(\text{compare with } \ (a, b) \to (c, d) = (a \to c, a \wedge d))$$

is an $\mathbf{N4}^{\perp}$-lattice.

2. For an $N4$-function $f : (X, X^1, \leq, \tau, g) \to (Y, Y^1, \leq', \tau', g')$, put

$$\mathsf{N}(f) := \mathsf{D}(f)$$

3. Categories $\mathcal{N}_4^*$ of $N4$-spaces and $N4$-functions and $\mathcal{N}_4$ of bounded **N4**-lattices and their homomorphisms are dually equivalent via N and the restriction of O.

$\mathcal{N}_3^*$ is a full subcategory of $\mathcal{N}_4^*$

1. A tuple $(X, \leq, \tau, g)$ is an $N3$-space if and only if $(X, X^+, \leq, \tau, g)$ is an $N4$-space