Hall subgroups and the pronormality

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Definition

A subgroup H of a group G is said to be *pronormal* if H and H^g are conjugate in $\langle H, H^g \rangle$ for every $g \in G$.

The notation "$H \operatorname{prn} G$" means "H is a pronormal subgroup of G".

Examples of pronormal subgroups

- Normal subgroups;
- Maximal subgroups;
- Sylow subgroups in finite groups;
- Sylow subgroups of normal subgroups in finite groups.

Definition

Let $p \in \mathbb{P}$. A subgroup P of a group G is called a *Sylow p-subgroup* if

- $|P|$ is a power of p (i. e., P is a *p-group*) while
- $|G : P|$ is not divisible by p.

Theorem (L. Sylow, 1872)

Let G be a finite group and let p be a prime. Then

$\mathcal{E}_p$ *G possesses a Sylow p-subgroup;*

$\mathcal{C}_p$ *every two Sylow p-subgroups are conjugate;*

$\mathcal{D}_p$ *every p-subgroup of G is included in a Sylow p-subgroup of G.*

We denote by $\mathrm{Syl}_p(G)$ the set of Sylow p-subgroup of a finite group G.

Corollary

Let G *be a finite group. Then* $P \operatorname{prn} G$ *for every* $P \in \mathrm{Syl}_p(G)$.

Corollary

Let G *be a finite group and* $A \trianglelefteq G$. *Then* $P \operatorname{prn} G$ *for every* $P \in \mathrm{Syl}_p(A)$.

Corollary (Frattini Argument)

Let G *be a finite group,* $A \trianglelefteq G$, *and* $P \in \mathrm{Syl}_p(A)$. *Then*

$$G = AN_G(P).$$

The Frattini Argument is closely connected with the pronormality.

If $A \trianglelefteq G$ and $H \leq A$ then
$G = AN_G(H)$ iff H and H^g are conjugate in A for every $g \in G$
(we write $H^A = H^G$, were $H^G = \{H^g \mid g \in G\}$).

As a consequence, if $H \operatorname{prn} G$ then $G = AN_G(H)$.

A natural generalization of the concept of a Sylow p-subgroup is that of a π-Hall subgroup.
We fix a set $\pi \subseteq \mathbb{P}$. Put $\pi' = \mathbb{P} \setminus \pi$.

Definition

A subgroup H of a finite group G is called a *π-Hall subgroup* if

- every prime divisor of $|H|$ belongs to π (H is a *π-subgroup*) and
- every prime divisor of $|G : H|$ belongs to π'.

The set of all π-Hall subgroups of G is denoted by $\mathrm{Hall}_\pi(G)$.

If $\pi = \{p\}$ then $\mathrm{Hall}_\pi(G) = \mathrm{Syl}_p(G)$.

What properties of Sylow subgroups hold for Hall subgroups?

What can one say about the pronormality of Hall subgroups?

Hall subgroups have some nice properties.

Proposition

Let G *be a finite group,* $A \trianglelefteq G$*, and* $H \in \mathrm{Hall}_\pi(G)$*. Then*

- $H \cap A \in \mathrm{Hall}_\pi(A)$ *and*
- $HA/A \in \mathrm{Hall}_\pi(G/A)$.

The analogue of Sylow's Theorem fails for Hall subgroups:

- In Alt_5 of order $60 = 2^2 \cdot 3 \cdot 5$, there are no elements and subgroups of order 15, hence Alt_5 does not have $\{3, 5\}$-Hall subgroups.
- In $\mathrm{GL}_3(2)$ of order $168 = 2^3 \cdot 3 \cdot 7$, there are exactly two conjugacy classes of subgroups of order $2^3 \cdot 3$ (= $\{2, 3\}$-Hall subgroups): the stabilizers of lines and planes, respectively.
- Every subgroup of order $12 = 2^2 \cdot 3$ (= a $\{2, 3\}$-Hall subgroup) of Alt_5 is a point stabilizer, and all point stabilizers are conjugate. On the other hand, Alt_5 includes a $\{2, 3\}$-subgroup $\langle (123), (12)(45) \rangle \simeq \mathrm{Sym}_3$ which acts without fixed points.

Theorem (P. Hall, 1928)

Let G be a solvable finite group and let π be a set of primes. Then

- $\mathcal{E}_\pi$ *G possesses a π-Hall subgroup;*
- $\mathcal{C}_\pi$ *every two π-Hall subgroups of G are conjugate;*
- $\mathcal{D}_\pi$ *every π-subgroup of G is included in a π-Hall subgroup of G.*

Corollary

Let G be a solvable group and $H \in \mathrm{Hall}_\pi(G)$. Then $H \,\mathrm{prn}\, G$.

Definition (P.Hall)

Given a set of primes π, we say that a finite group G *satisfies*

- $\mathcal{E}_\pi$ if $\mathrm{Hall}_\pi(G) \neq \varnothing$ (i.e., there exists a π-Hall subgroup in G);
- $\mathcal{C}_\pi$ if G satisfies $\mathcal{E}_\pi$ and every two π-Hall subgroups of G are conjugate;
- $\mathcal{D}_\pi$ if G satisfies $\mathcal{C}_\pi$ and every π-subgroup of G is included in a π-Hall subgroup.

A group G satisfying $\mathcal{E}_\pi$ (resp., $\mathcal{C}_\pi$, $\mathcal{D}_\pi$) is called an $\mathcal{E}_\pi$- (resp., $\mathcal{C}_\pi$-, $\mathcal{D}_\pi$-) *group*.

Given a set of primes π, we also denote by $\mathcal{E}_\pi$, $\mathcal{C}_\pi$, and $\mathcal{D}_\pi$ the classes of all finite $\mathcal{E}_\pi$-, $\mathcal{C}_\pi$-, and $\mathcal{D}_\pi$- groups, respectively.

There exist sets $\pi \subseteq \mathbb{P}$ such that $\mathcal{E}_\pi = \mathcal{C}_\pi$.

Theorem (F.Gross, 1987)

If $2 \notin \pi$ then $\mathcal{E}_\pi = \mathcal{C}_\pi$.

Proposition

If $\mathcal{E}_\pi = \mathcal{C}_\pi$ for some π then π-Hall subgroups of finite groups are pronormal.

If $H \in \mathrm{Hall}_\pi(G)$ then, given $g \in G$, we have $\langle H, H^g \rangle \in \mathcal{E}_\pi = \mathcal{C}_\pi$, and hence H and H^g are conjugate in $\langle H, H^g \rangle$.

Corollary

Hall subgroups of odd order are pronormal.

Let $\pi \subseteq \mathbb{P}$ and $\mathcal{E}_\pi \neq \mathcal{C}_\pi$.
Take $X \in \mathcal{E}_\pi \setminus \mathcal{C}_\pi$ and non-conjugate $U, V \in \mathrm{Hall}_\pi(X)$.
Let $n \in \pi'$. Consider

$$Y = \underbrace{X \times X \times \cdots \times X \times X}_{n \text{ times}}$$

and $\tau \in \mathrm{Aut}(Y)$, where

$$\tau : (x_1, x_2, \dots, x_{n-1}, x_n) \mapsto (x_2, x_3, \dots, x_n, x_1)$$

for all $x_1, x_2, \dots, x_{n-1}, x_n \in X$.
Let $G = Y \rtimes \langle \tau \rangle \simeq X \wr \mathbb{Z}_n$. The subgroups

$$H = V \times \underbrace{U \times \cdots \times U \times U}_{n-1 \text{ times}},$$

$$K = \underbrace{U \times U \times \cdots \times U}_{n-1 \text{ times}} \times V$$

of Y are π-Hall in Y and in G. Moreover, $H^\tau = K$. But H and K are not conjugate in Y, and hence are non-pronormal in G.

In the talk we are concerned with the following problems:

- What can one say about $\mathcal{E}_\pi$-groups in which there exists a pronormal π-Hall subgroup?

  (the results are obtained in collaboration with Prof. E.P.Vdovin)

- What can one say about $\mathcal{E}_\pi$-groups in which every π-Hall subgroup is pronormal?

  (the results are obtained in collaboration with Prof. Wenbin Guo)

On the existence of pronormal subgroups in $\mathcal{E}_\pi$-groups

in collaboration with Prof. E.P.Vdovin

Theorem (E.Vdovin, D.R., 2013)

In $\mathcal{C}_\pi$-groups, every π-Hall subgroup is pronormal.

This theorem is equivalent to the following statement.

Theorem (E.Vdovin, D.R., 2013)

Let $G \in \mathcal{C}_\pi$, $H \in \mathrm{Hall}_\pi(G)$, and $H \leqslant M \leqslant G$. Then $M \in \mathcal{C}_\pi$.

In order to prove these theorems, the classification of finite simple groups and the classification of Hall subgroups in such groups are used.

Main ingredients of our proof:

- The theorem on the number of classes of conjugate π-Hall subgroups in finite simple groups (the Class Number Theorem);
- The converse to Gross' theorem on the existence of π-Hall subgroups;
- The theorem on the pronormality of Hall subgroups of finite simple groups.

The same ingredients are sufficient to prove a statement stronger than the pronormality of π-Hall subgroups in $\mathcal{C}_\pi$-groups.

Theorem 1 (E.Vdovin, D.R., new)

If $G \in \mathcal{E}_\pi$ then G has a pronormal π-Hall subgroup.

Thus, the class of groups containing a pronormal π-Hall subgroup coincides with $\mathcal{E}_\pi$.

Theorem 2 (E.Vdovin, D.R., new)

If $G \in \mathcal{E}_\pi$ and $A \trianglelefteq G$ then there exists $H \in \mathrm{Hall}_\pi(A)$ such that

$$H \operatorname{prn} G.$$

Theorem 3 (Frattini Argument, E.Vdovin, D.R., new)

If $G \in \mathcal{E}_\pi$ and $A \trianglelefteq G$ then there exists $H \in \mathrm{Hall}_\pi(A)$ such that

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If $G \in \mathcal{E}_\pi$ *and* $A \trianglelefteq G$ *then* $G = AN_G(H)$ *for some* $H \in \mathrm{Hall}_\pi(A)$.

Corollary (E.Vdovin, D.R., 2006)

If $G \in \mathcal{D}_\pi$ *and* $A \trianglelefteq G$ *then* $A \in \mathcal{D}_\pi$.

Corollary (E.Vdovin, D.R., 2010)

If $G \in \mathcal{C}_\pi$, $H \in \mathrm{Hall}_\pi(G)$, *and* $A \trianglelefteq G$ *then* $HA \in \mathcal{C}_\pi$.

Corollary (E.Vdovin, D.R., 2011)

If $G \in \mathcal{E}_\pi$ *and* $A \trianglelefteq G$ *then* $\mathrm{Hall}_\pi(G/A) = \{HA/A \mid H \in \mathrm{Hall}_\pi(G)\}$.

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Let $A \trianglelefteq G$. *Then* $G \in \mathcal{E}_\pi$ *iff* $A \in \mathcal{E}_\pi$, $G/A \in \mathcal{E}_\pi$, *and there is* $H \in \mathrm{Hall}_\pi(A)$ *such that* $H^A = H^G$.

Corollary (E.Vdovin, D.R., new)

Let $G \in \mathcal{E}_\pi$, $A \leq \mathrm{Aut}(G)$ *and* $(|G|, |A|) = 1$. *Then there exists an* A*-invariant* $H \in \mathrm{Hall}_\pi(G)$.

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Compare these statements with their analogs for Sylow subgroups.

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Compare these statements with their analogs for Sylow subgroups.

Corollary 1 to Sylow's Theorem

If G is a group then $P \operatorname{prn} G$ for every $P \in \mathrm{Syl}_p(G)$.

Corollary 2 to Sylow's Theorem

If G is a group and $A \trianglelefteq G$ then $P \operatorname{prn} G$ for every $P \in \mathrm{Syl}_p(A)$.

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Let $n \in \pi'$. Consider

$$Y = \underbrace{X \times X \times \cdots \times X \times X}_{n \text{ times}}$$

and $\tau \in \mathrm{Aut}(Y)$, where

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$$H = V \times \underbrace{U \times \cdots \times U \times U}_{n-1 \text{ times}},$$

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of Y are π-Hall in Y and in G. Moreover, $H^\tau = K$.

Thus, in Theorems 1–3, one cannot replace "for some" with "for every".

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Let $\pi = \{2,3\}$ and let $A = \mathrm{GL}_3(2)$. Then A has exactly two classes of conjugate π-Hall subgroups with the representatives

$$U = \begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & * \\ & \boxed{1} \end{pmatrix} \text{ and } V = \begin{pmatrix} \boxed{1} & * \\ & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}.$$

The first one consists of the line stabilizers in the natural representation of A, and the second one consists of the plane stabilizers.

Consider the automorphism $\iota : x \mapsto (x^t)^{-1}$, $x \in A$, of A (here x^t is the transpose of x). Then $|\iota| = 2$ and ι interchanges U^A and V^A.

Let $G = A \rtimes \langle \iota \rangle$. Then $K^A \neq K^G$ for every $K \in \mathrm{Hall}_\pi(A)$. In particular, K is non-pronormal in G and $G \neq AN_G(K)$.

Suppose, $H \in \mathrm{Hall}_\pi(G)$. Then $G = HA$ and G stabilizes $(H \cap A)^A$. But $(H \cap A)^A$ coincides either with U^A or with V^A, and hence G does not stabilize $(H \cap A)^A$. A contradiction.

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Suppose, $H \in \mathrm{Hall}_\pi(G)$. Then $G = HA$ and G stabilizes $(H \cap A)^A$. But $(H \cap A)^A$ coincides either with U^A or with V^A, and hence G does not stabilize $(H \cap A)^A$. A contradiction.

Thus, $G \notin \mathcal{E}_\pi$.

Thus, in Theorems 2 and 3, one cannot replace "$G \in \mathcal{E}_\pi$" with "$A \in \mathcal{E}_\pi$".

In our proof of this Theorem, we use the classification of finite simple groups and the classification of Hall subgroups in such groups.
If G is a finite group then we denote by $k_\pi(G)$ the number of classes of conjugate π-Hall subgroups of G.

Class Number Theorem (E.Vdovin and D.R., 2010, mod CFSG)

Let $S \in \mathcal{E}_\pi$ be a finite simple group. Then the following statements hold:

- *if $2 \notin \pi$, then $k_\pi(S) = 1$;*
- *if $3 \notin \pi$, then $k_\pi(S) \in \{1, 2\}$;*
- *if $2, 3 \in \pi$, then $k_\pi(S) \in \{1, 2, 3, 4, 9\}$.*

In particular, $k_\pi(S)$ is a bounded π-number.

Furthermore, if $S \trianglelefteq G$ then we denote by $k_\pi^G(S)$ the number of classes of conjugate π-Hall subgroups K of S such that $K = H \cap S$ for some $H \in \mathrm{Hall}_\pi(G)$.
As a consequence of the Class Number Theorem, we have

CNT-Corollary

Let $G \in \mathcal{E}_\pi$ possess a unique minimal normal subgroup S and let S be a finite simple group. Then the following statements hold:

- *if $2 \notin \pi$, then $k_\pi^G(S) = 1$;*
- *if $3 \notin \pi$, then $k_\pi^G(S) \in \{1, 2\}$;*
- *if $2, 3 \in \pi$, then $k_\pi^G(S) \in \{1, 2, 3, 4, 9\}$.*

In particular, $k_\pi^G(S)$ is a π-number.

If $G \in \mathcal{E}_\pi$ *and* $A \trianglelefteq G$ *then* $H^G = H^A$ *for some* $H \in \mathrm{Hall}_\pi(A)$.

Firstly, let A be the unique minimal normal subgroup of G and let A be simple.

Denote by Ω the set of classes of conjugate π-Hall subgroups K of S such that $K = H \cap A$ for some $H \in \mathrm{Hall}_\pi(G)$.

Then $|\Omega| = k_\pi^G(A)$ and G acts on Ω in a natural way.

Consider

$$H \in \mathrm{Hall}_\pi(G) \text{ and } \mathcal{K} = (H \cap A)^A = \{H^a \cap A \mid a \in A\} \in \Omega.$$

Since $H \cap A \trianglelefteq H$, H leaves invariant $\mathcal{K}$, and hence $H \leqslant G_{\mathcal{K}}$.

Thus, $|\mathcal{K}^G|$ divides $|G : H|$ and $|\mathcal{K}^G|$ is a π'-number.

On the other hand, $|\mathcal{K}^G| \leq |\Omega| = k_\pi^G(A)$. In view of CNT-Corollary, one of the following statements holds:

- $|\mathcal{K}^G|$ is a π-number;
- $2, 3 \in \pi$, $k_\pi^G(A) = 9$, and $|\mathcal{K}^G| \in \{5, 7\}$.

In the first case, $|\mathcal{K}^G| = 1$, i. e. $(H \cap A)^A = (H \cap A)^G$.

In the second case, the power of every orbit $\mathcal{L}^G$ of G on Ω that differs from $\mathcal{K}^G$ is at most 4. Hence $|\mathcal{L}^G|$ is a π-number and $|\mathcal{L}^G| = 1$.

Now consider the situation where A is a minimal normal subgroup (A is not necessarily simple and G does not necessarily have a unique minimal normal subgroup).

Definition

Suppose A, B, H are subgroups of G such that $B \trianglelefteq A$.
Then $N_H(A/B) = N_H(A) \cap N_H(B)$ is the *normalizer* of A/B in H.
If $x \in N_H(A/B)$ then x induces an automorphism of A/B by

$$Ba \mapsto Bx^{-1}ax.$$

Thus there exists a homomorphism $N_H(A/B) \to \mathrm{Aut}(A/B)$.
The image of $N_H(A/B)$ under this homomorphism is denoted by $\mathrm{Aut}_H(A/B)$ and is called the *group of H-induced automorphisms* of A/B.

Theorem (F.Gross, 1986)

Let $1 = G_0 < G_1 < \ldots < G_n = G$ be a composition series for a finite group G which is a refinement of a chief series for G. If $\mathrm{Aut}_G(G_i/G_{i-1}) \in \mathcal{E}_\pi$ for all $i = 1, \ldots, n$, then $G \in \mathcal{E}_\pi$.

Theorem (E.Vdovin and D.R., 2011)

Let $1 = G_0 < G_1 < \ldots < G_n = G$ be a composition series for a finite group G. If $G \in \mathcal{E}_\pi$ then $\mathrm{Aut}_G(G_i/G_{i-1}) \in \mathcal{E}_\pi$ for all i.

In the considered situation, $A = S_1 \times \cdots \times S_n$,
where S_i are simple and $S_i = S_1^{g_i}$ for some $g_i \in G$, $i = 1, \dots, n$.

Then $g_1, \dots, g_n$ is a right transversal for $N_G(S_1)$ in G.

By the converse to Gross' Theorem,
$\mathrm{Aut}_G(S_1) = N_G(S_1)/C_G(S_1) \in \mathcal{E}_\pi$.

By the above-considered case, $S_1 \cong \mathrm{Inn}(S_1)$ includes an $\mathrm{Aut}_G(S_1)$-invariant conjugacy class $\mathcal{K}$ of π-Hall subgroups.

Let $U \in \mathcal{K}$. Then

- $U^{g_i} \in \mathrm{Hall}_\pi(S_i)$,
- $V = \langle U^{g_i} \mid i = 1, \dots, n \rangle \in \mathrm{Hall}_\pi(A)$, and
- $V^G = V^A$, i.e., $G = AN_G(V)$.

Now, one can investigate the general situation in Theorem 3 by using induction on $|G|$.

Theorem 1: *In an* $\mathcal{E}_\pi$*-group* G, $H \operatorname{prn} G$ *for some* $H \in \mathrm{Hall}_\pi(G)$.

Theorem (E.Vdovin, D.R., 2012)

In the finite simple groups, the Hall subgroups are pronormal.

Thus, we can assume that G is not simple. Let A be a minimal normal subgroup of G. Then $A = S_1 \times \cdots \times S_n$, where S_i are simple. Moreover, $K \in \mathrm{Hall}_\pi(A) \Rightarrow K \operatorname{prn} A$.
By Theorem 1, $G = AN_G(K)$ for some $K \in \mathrm{Hall}_\pi(A)$.
By induction, $H/K \operatorname{prn} N_G(K)/K$ for some $H/K \in \mathrm{Hall}_\pi(N_G(K)/K)$ and $H \in \mathrm{Hall}_\pi(G)$.

$$H/K \operatorname{prn} N_G(K)/K \Rightarrow HA/A \operatorname{prn} AN_G(K)/A = G/A.$$

Proposition

Let $H \in \mathrm{Hall}_\pi(G)$, $A \trianglelefteq G$, *and*

- $HA/H \operatorname{prn} G/A$;
- $H \cap A \operatorname{prn} A$;
- $(H \cap A)^A = (H \cap A)^G$.

Then $H \operatorname{prn} G$.

Conjecture 1

Every Hall subgroup is pronormal in its normal closure.

If $H \in \mathrm{Hall}_\pi(G)$ then $\langle H^G \rangle = O^{\pi'}(G)$.

Conjecture 1'

If $H \in \mathrm{Hall}_\pi(G)$ and $\langle H^G \rangle = G$ then $H \,\mathrm{prn}\, G$.

A subgroup H of G is said to be *strongly pronormal* if K^g is conjugate to a subgroup of H in $\langle H, K^g \rangle$ for every $g \in G$ and $K \leq H$.

Conjecture 2

Every pronormal Hall subgroup is strongly pronormal.

Conjecture 3

Every $\mathcal{E}_\pi$-group contains a strongly pronormal π-Hall subgroup.

On $\mathcal{E}_\pi$-groups in which every π-Hall subgroup is pronormal

In collaboration with Prof. Guo Wenbin

By analogy with P.Hall's notation, we will say that a group G *satisfies* $\mathcal{P}_\pi$ (is a $\mathcal{P}_\pi$*-group*, belongs to the *class* $\mathcal{P}_\pi$) if $G \in \mathcal{E}_\pi$ and every π-Hall subgroups in G is pronormal.

Theorem

(1) *If* $G \in \mathcal{E}_\pi$ *is simple then* $G \in \mathcal{P}_\pi$.

(2) $\mathcal{C}_\pi \subseteq \mathcal{P}_\pi \subseteq \mathcal{E}_\pi$.

(3) *If* $\mathcal{C}_\pi \subset \mathcal{E}_\pi$ *then* $\mathcal{C}_\pi \subset \mathcal{P}_\pi \subset \mathcal{E}_\pi$.

$\mathrm{S}\mathfrak{X} = \{G \mid G$ is isomorphic to a subgroup of $H \in \mathfrak{X}\}$;

$\mathrm{Q}\mathfrak{X} = \{G \mid G$ is an epimorphic image of $H \in \mathfrak{X}\}$;

$\mathrm{S}_n\mathfrak{X} = \{G \mid G$ is isomorphic to a subnormal subgroup of $H \in \mathfrak{X}\}$

$\mathrm{R}_0\mathfrak{X} = \{G \mid \exists N_i \trianglelefteq G \ (i = 1, \dots, m)$ with $G/N_i \in \mathfrak{X}$ and $\bigcap\limits_{i=1}^{m} N_i = 1\}$;

$\mathrm{N}_0\mathfrak{X} = \{G \mid \exists N_i \trianglelefteq\trianglelefteq G \ (i = 1, \dots, m)$ with $N_i \in \mathfrak{X}$ and $G = \langle N_1, \dots, N_m \rangle\}$;

$\mathrm{E}\mathfrak{X} = \{G \mid G$ possesses a series $1 = G_0 \trianglelefteq G_1 \trianglelefteq \dots \trianglelefteq G_m = G$ with $G_i/G_{i-1} \in \mathfrak{X} \ (i = 1, \dots, m)\}$.

If $\pi = \{3, 5\}$ then the classes $\mathcal{D}_\pi$, $\mathcal{C}_\pi$, $\mathcal{P}_\pi$, and $\mathcal{E}_\pi$ are not S-closed: $\mathrm{SL}_2(16) \in \mathcal{D}_\pi$ while $\mathrm{SL}_2(4) \cong \mathrm{Alt}_5 \notin \mathcal{E}_\pi$.

Table: Is it true that $\mathrm{C}\mathfrak{X} = \mathfrak{X}$, $\mathfrak{X} \in \{\mathcal{D}_\pi, \mathcal{E}_\pi, \mathcal{C}_\pi, \mathcal{P}_\pi\}$?

C	$\mathcal{D}_\pi$	$\mathcal{E}_\pi$	$\mathcal{C}_\pi$	$\mathcal{P}_\pi$
S	no	no	no	no
Q	yes	yes	yes	yes
S_n	yes	yes	no	no
R_0	yes	yes	yes	yes
N_0	yes	no	yes	no
E	yes	no	yes	no

Theorem (Wenbin Guo, D.R., mod CFSG)

The following statements hold:

(A) $\mathrm{C}\mathcal{P}_\pi = \mathcal{P}_\pi$ *for every set* π *of primes and* $\mathrm{C} \in \{\mathrm{Q}, \mathrm{R}_0\}$.

(B) *If* $\mathrm{C} \in \{\mathrm{S}, \mathrm{S}_n, \mathrm{N}_0, \mathrm{E}\}$, *then* $\mathrm{C}\mathcal{P}_\pi \neq \mathcal{P}_\pi$ *for a set* π *of primes.*

(C) *If* $\mathrm{C} \in \{\mathrm{S}_n, \mathrm{E}\}$ *and* $\mathrm{C}\mathcal{P}_\pi = \mathcal{P}_\pi$ *for a set* π *of primes then* $\mathrm{C}\mathcal{E}_\pi = \mathcal{E}_\pi$ *and* $\mathrm{C}\mathcal{C}_\pi = \mathcal{C}_\pi$.

Corollary (Wenbin Guo, D.R., mod CFSG)

Let $\mathrm{C} \in \{\mathrm{S}, \mathrm{Q}, \mathrm{S}_n, \mathrm{R}_0, \mathrm{N}_0, \mathrm{E}\}$. *Then the following statements are equivalent:*

(1) $\mathrm{C}\mathcal{P}_\pi = \mathcal{P}_\pi$ *for every set* π *of primes;*

(2) $\mathrm{C}\mathcal{E}_\pi = \mathcal{E}_\pi$ *and* $\mathrm{C}\mathcal{C}_\pi = \mathcal{C}_\pi$ *for every set* π *of primes.*

Corollary (Wenbin Guo, D.R., mod CFSG)

For a set π *of primes,* $\mathcal{P}_\pi$ *is a saturated formation.*

Corollary (Wenbin Guo, D.R.)

There exist sets π *(for example,* $\pi = \{2, 3\}$*) of primes such that* $\mathcal{P}_\pi$ *is not a Fitting class.*

Proposition (mod CFSG)

If $\mathcal{E}_\pi = \mathcal{C}_\pi$ *and* $\mathrm{C} \in \{\mathrm{Q}, \mathrm{S}_n, \mathrm{R}_0, \mathrm{N}_0, \mathrm{E}\}$ *then* $\mathrm{C}\mathcal{P}_\pi = \mathcal{P}_\pi$.
In particular, $\mathcal{P}_\pi$ *is a Fitting class.*

Thank you for your attention!