Positive provability logic

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Strictly positive modal formulas

The language of modal logic extends that propositional calculus by a family of unary connectives $\{\Diamond_i : i \in I\}$.

Strictly positive modal formulas are defined by the grammar:

$$A ::= p \mid \top \mid (A \wedge B) \mid \Diamond_i A, \quad i \in I.$$

We are interested in the implications $A \to B$ where A and B are strictly positive.

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The interest towards strictly positive (fragments of) modal logics independently emerged around 2010 in two different disciplines:

- The work on proof-theoretic applications of *provability logic* by Beklemishev, Dashkov, et al.;
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Advantages

- Strictly positive fragment of a modal logic is (in the considered cases) *much simpler* than the original logic.
- Typically, strictly positive fragments of standard modal logics are polytime decidable.

On the other hand, the language is *sufficiently expressive*:

- Main applications of provability logic to the analysis of arithmetical theories can be stated in terms of strictly positive logics.
- Strictly positive logics allow for alternative arithmetical interpretations that are quite natural from a proof-theoretic point of view.

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Provability logic

- S a gödelian theory (r.e. theory containing enough arithmetic)
- $\Box_S(x)$ provability predicate in S
- σ a substitution of S-sentences for propositional variables
- $\varphi \longmapsto \varphi^\sigma$ arithmetical translation of φ under σ
  $(\Box\varphi)^\sigma = \Box_S(\ulcorner \varphi^\sigma \urcorner)$.

Def. $PL(S) := \{\varphi : \forall\sigma\ S \vdash \varphi^\sigma\}$ the provability logic of S.

Th. (Solovay 76) $PL(S) = \mathrm{GL}$ if $\mathbb{N} \vDash S$.

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Gödel–Löb Logic GL

Axioms of GL:

- Tautologies;
- $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$;
- $\Box(\Box\varphi \to \varphi) \to \Box\varphi$.

Rules: modus ponens, $\varphi/\Box\varphi$.

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Shift of emphasis

From arithmetical completeness results to other types of questions:

- stronger provability-like concepts (reflection principles);
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Reflection principles

Emerged in the 1930s in the work of Rosser, Kleene and Turing. Later (in the 1960s) it was taken up by Kreisel, Levy, Feferman and others.

Cf. G. Kreisel and A. Levy (1968): *Reflection principles and their use for establishing the complexity of axiomatic systems.*

Reflection principles

Notation:

$\Box_S(\varphi)$ ‘φ *is provable in* S’

$Tr_n(\sigma)$ ‘σ *is the Gödel number of a true* Σ_n*-sentence*’

Restricted reflection principles:

$R_0(S)$ $\mathrm{Con}(S)$

$R_n(S)$ $\forall \sigma \in \Sigma_n\, (\Box_S \sigma \to Tr_n(\sigma))$, for $n \geq 1$.

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Unrestricted reflection

- Uniform reflection: $R_\omega(S) := \{R_n(S) : n \in \omega\}$.

  $\{\forall x\, (\Box_S \varphi(\dot{x}) \to \varphi(x)) : \varphi(x) \text{ any arithmetical formula}\}$

- Local reflection:

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Reflection algebra of S

Let $\mathcal{L}_S$ be the Lindenbaum–Tarski boolean algebra of S sentences.

- Each R_n correctly defines an operator on the equivalence classes of $\mathcal{L}_S$: $\langle n \rangle : [\varphi] \longmapsto [R_n(S + \varphi)]$.
- The algebra $(\mathcal{L}_S, \langle 0 \rangle, \langle 1 \rangle, \dots)$ is the *reflection algebra of S*.

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GLP: a Hilbert-style calculus

Basic symbols are now $[n]$, for each $n \in \omega$, and $\langle n \rangle$ is treated as an abbreviation: $\langle n \rangle \varphi = \neg [n] \neg \varphi$.

1. Tautologies;
2. $[n](\varphi \to \psi) \to ([n]\varphi \to [n]\psi)$;
3. $[n]([n]\varphi \to \varphi) \to [n]\varphi$;
4. $[n]\varphi \to [n+1]\varphi$;
5. $\langle n \rangle \varphi \to [n+1]\langle n \rangle \varphi$

Rules: modus ponens, $\varphi \vdash [n]\varphi$.

Th. (Japaridze, 1986) GLP $\vdash \varphi(\vec{x})$ iff $\mathcal{L}_S \models \forall \vec{x} \, (\varphi(\vec{x}) = 1)$, provided $\mathbb{N} \models S$.

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Reflection calculus RC

Language: $\alpha ::= \top \mid p \mid (\alpha_1 \wedge \alpha_2) \mid n\alpha \qquad n \in \omega$
Example: $\alpha = 3(2p \wedge 32\top)$, or shortly: $3(2p \wedge 32)$.

Sequents: $\alpha \vdash \beta$.

RC rules:

1. $\alpha \vdash \alpha$; $\quad \alpha \vdash \top$; $\quad$ if $\alpha \vdash \beta$ and $\beta \vdash \gamma$ then $\alpha \vdash \gamma$;
2. $\alpha \wedge \beta \vdash \alpha, \beta$; $\quad$ if $\alpha \vdash \beta$ and $\alpha \vdash \gamma$ then $\alpha \vdash \beta \wedge \gamma$;
3. $nn\alpha \vdash n\alpha$; $\quad$ if $\alpha \vdash \beta$ then $n\alpha \vdash n\beta$;
4. $n\alpha \vdash m\alpha$ for $n > m$;
5. $n\alpha \wedge m\beta \vdash n(\alpha \wedge m\beta)$ for $n > m$.

Ex. $3 \wedge 23 \vdash 3(\top \wedge 23) \vdash 323$.

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Interpretation of RC in GLP

- RC can be seen as a strictly positive fragment of GLP.
- Interpretation: $3(2p \wedge 32\top) \mapsto \langle 3\rangle(\langle 2\rangle p \wedge \langle 3\rangle\langle 2\rangle\top)$

Theorems (E. Dashkov).

1. GLP is a conservative extension of RC;
2. RC is polytime decidable;
3. RC enjoys the finite model property.

RC^0 *as an ordinal notation system*

Let RC^0 denote the variable-free fragment of RC.
Let W denote the set of all RC^0-formulas. For $\alpha, \beta \in W$ define:

- $\alpha \sim \beta$ if $\alpha \vdash \beta$ and $\beta \vdash \alpha$ in RC^0;
- $\alpha <_n \beta$ if $\beta \vdash n\alpha$.

Theorem.

1. Every $\alpha \in W$ is equivalent to a *word* (formula without $\wedge$);
2. $(W/{\sim}, <_0)$ is isomorphic to $(\varepsilon_0, <)$.

Rem. $\varepsilon_0 = \sup\{\omega, \omega^\omega, \omega^{\omega^\omega}, \dots\}$ is the characteristic ordinal of Peano arithmetic.

The system $RC\omega$

The language of RC is extended by a new modality symbol ω. The rules of $RC\omega$ are:

- The rules of RC stated for all $n \leq \omega$;
- $\omega\alpha \vdash \alpha$.

The intended interpretation of ω is R_ω. Hence, the strictly positive modal formulas should now be understood as (possibly infinite) *sets* of arithmetical sentences.

Axiom $\omega\alpha \vdash \alpha$

- Modal logically, this schema means $\Diamond A \to A$, i.e., $A \to \Box A$.
- Notice that in the context of classical modal logic the principle $A \to \Box A$ only has trivial (discrete) Kripke frames.
- Not so in the strictly positive logic! The same principle also plays a certain role in intuitionistic modal logic.

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Arithmetical interpretation of RCω

Let S be a gödelian theory.

- An *arithmetical substitution* σ assigns to each variable a gödelian theory extending S (i.e. a primitive recursive set of sentences together with a p.r. formula defining this set).
- *Interpretation* α^σ of a strictly positive formula α in S:
  - $\top^\sigma = \varnothing$; $(\alpha \wedge \beta)^\sigma = (\alpha^\sigma \cup \beta^\sigma)$;
  - $(n\alpha)^\sigma = \{R_n(S + \alpha^\sigma)\}$;
  - $(\omega\alpha)^\sigma = R_\omega(S + \alpha^\sigma)$.

Arithmetical completeness

Suppose $\mathbb{N} \vDash S$.
Theorem 1. $\alpha \vdash \beta$ in $RC\omega$ iff $\forall \sigma\ S + \alpha^\sigma \vdash \beta^\sigma$.

Theorem 2. $RC\omega$ enjoys the finite model property and is polytime decidable.

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Example

- A simple Kripke model shows that

$$\omega p \wedge \omega q \nvdash_{RC\omega} \omega(p \wedge q).$$

By Theorem 1, there are arithmetical theories P, Q such that

$$S + R_\omega(P) + R_\omega(Q) \nvdash R_\omega(P + Q).$$

- It is easy to see that any such pair must have unrestricted arithmetical complexity over S.

$RC^0\omega$ *as an ordinal notation system*

Let $RC^0\omega$ denote the variable-free fragment of $\mathrm{RC}\omega$.
Let W_ω denote the set of all $RC^0\omega$-formulas.

Theorem 3. In $RC^0\omega$,

1. Every $\alpha \in W_\omega$ is equivalent to a *word* (formula without $\wedge$);
2. $(W_\omega/{\sim}, <_0)$ is isomorphic to $(\varepsilon_\omega, <)$.

Here ε_ω is the ω-th ordinal α such that $\omega^\alpha = \alpha$.

Open questions

- Study systematically strictly positive normal logics (*s.p.l.*). State general sufficient conditions guaranteeing nice semantic properties of such logics.
- Characterize strictly positive fragments of standard polymodal logics by positive calculi. (Some preliminary work done by Dashkov, Kaniskin and more to come.)
- Study the complexity of s.p.l.. (Need not always be polytime decidable or even decidable.)
- Study the analogy between s.p.l. and semi-Thue systems.
- Positive Craig interpolation?

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Papers

- E. Dashkov. *On the positive fragment of the polymodal provability logic GLP.* Mathematical Notes, 2012, Vol. 91, No. 3, pp. 318–333.
- L. Beklemishev. *Positive provability logic for uniform reflection principles.* Annals of Pure and Applied Logic, published online August 2013. DOI: 10.1016/j.apal.2013.07.006