On interpolation in n-transitive modal logics

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The minimal modal logic K

The minimal modal logic K is determined by a set of its axioms and rules of inference:

Axioms:

1. All the tautologies of the classical two-valued logic;
2. $\Box(p \to q) \to (\Box p \to \Box q)$, $\Diamond p \equiv \neg\Box\neg p$.

Rules of inference:

R1. $\dfrac{A, (A \to B)}{B}$ (modus ponens);

R2. $\dfrac{A}{\Box A}$ (necessitation rule);

R3. Substitution (of formulas for variables).

Normal modal logics

We use standard denotation for some members of $NE(K)$:

$K^n = [n]p \to \Box^{n+1}p$, where $[n]p = p\&\Box p\& \ldots \&\Box^n p$.

$wK4 = K + ((p\&\Box p) \to \Box\Box p)$;

$K4 = K + (\Box p \to \Box\Box p)$;

$DL = wK4 + (p \to \Box\Diamond p)$;

$S4 = K4 + (\Box p \to p)$;

$S5 = S4 + (p \to \Box\Diamond p)$.

Interpolation properties

If $\mathbf{p}$ is a list of propositional variables, let $A(\mathbf{p})$ denote a formula whose all variables are in $\mathbf{p}$. Suppose that $\mathbf{p}, \mathbf{q}, \mathbf{r}$ are disjoint lists of propositional variables.

CIP: If $\vdash_L A(\mathbf{p}, \mathbf{q}) \to B(\mathbf{p}, \mathbf{r})$, then there exist a formula $C(\mathbf{p})$ such that $\vdash_L A(\mathbf{p}, \mathbf{q}) \to C(\mathbf{p})$ and $\vdash_L C(\mathbf{p}) \to B(\mathbf{p}, r)$.

IPD: If $A(\mathbf{p}, \mathbf{q}) \vdash_L B(\mathbf{p}, \mathbf{r})$, then there exist a formula $C(\mathbf{p})$ such that $A(\mathbf{p}, \mathbf{q}) \vdash_L C(\mathbf{p})$ and $C(\mathbf{p}) \vdash_L B(\mathbf{p}, \mathbf{r})$.

WIP: If $A(\mathbf{p}, \mathbf{q}),\ B(\mathbf{p}, \mathbf{r}) \vdash_L \bot$, then there exist a formula $C(\mathbf{p})$ such that $A(\mathbf{p}, \mathbf{q}) \vdash_L C(\mathbf{p})$ and $C(\mathbf{p}),\ B(\mathbf{p}, r) \vdash_L \bot$.

Facts

$CIP \Rightarrow IPD \Rightarrow WIP$ (in normal modal logics reverse arrow fail)

L. Maksimova: *CIP*, *IPD* and *WIP* are equivalent over $S5$.

L. Maksimova: *CIP* and *IPD* are decidable over $S4$.

A. Chagrov: *CIP* and *IPD* are undecidable over $K4$.

A. Karpenko: *WIP* is decidable over $K4$.

Facts

Larisa L. Maksimova proved that *IPD* is equivalent to amalgamation of correspondence varieties of modal algebras.

For a given logic L, its associated variety $V(L) = \{\mathfrak{A} \mid \mathfrak{A} \models L\}$. It is known that $L = \{A \mid (\forall \mathfrak{A} \in V(L))(\mathfrak{A} \models A)\}$. For every class K of modal algebras the set $L(K) = \{A \mid (\forall \mathfrak{A} \in K)(\mathfrak{A} \models A)\}$ is a modal logic.

A class V has the *amalgamation property* if it satisfies the following condition:

AP: For any $\mathfrak{B}, \mathfrak{C} \in V$ with a common subalgebra $\mathfrak{A}$, there exist an algebra $\mathfrak{D}$ in V and monomorphisms $\delta : \mathfrak{B} \to \mathfrak{D}$ and $\varepsilon : \mathfrak{C} \to \mathfrak{D}$ such that $\delta(x) = \varepsilon(x)$ for all $x \in \mathcal{A}$.

Modal algebras

A *modal algebra* is an algebra $\mathfrak{A} = (|\mathfrak{A}|, \to, \bot, \Box)$ which satisfies the identities of Boolean algebras for $\to$ and $\bot$ and, moreover, the conditions $\Box\top = \top$, where $\top = \bot \to \bot$, and $\Box(x \to y) \leq \Box x \to \Box y$.

A modal algebra $\mathfrak{A}$ is *weak transitive or wK4-algebra* if it satisfies $x \& \Box x \leq \Box\Box x$.

A modal algebra $\mathfrak{A}$ is *n-transitive* if it satisfies $[n]x \leq \Box^{n+1}x$, where $[n]x = x \& \Box x \& \ldots \& \Box^n x$.

A weak transitive modal algebra is called *DL-algebra* if it satisfies $x \leq \Box\Diamond x$, where $\Diamond x = \neg\Box\neg x$.

Important theorems

Theorem [Maksimova] : For any normal modal logic L the following are equivalent:

1. L has IPD;
2. the variety $V(L)$ has AP;
3. for a given finitely generated finitely indecomposable algebras $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ in $V(L)$ and monomorphisms $\beta : \mathfrak{A} \to \mathfrak{B}$ and $\gamma : \mathfrak{A} \to \mathfrak{C}$ there exist an algebra $\mathfrak{D}$ in $V(L)$ and monomorphisms $\delta : \mathfrak{B} \to \mathfrak{D}$, $\varepsilon : \mathfrak{C} \to \mathfrak{D}$, with $\delta\beta = \varepsilon\gamma$.

Theorem [Maksimova] : For any normal modal logic L the following are equivalent:

1. L has WIP;
2. the class $Sim(V(L))$ of simple algebras of $V(L)$ has AP;
3. the class $FG(Sim(V(L)))$ of finitely generated and simple algebras of $V(L)$ has AP.

Simple $wK4$ algebras

An algebra is said to be *simple* if it has exactly two congruences.

Theorem [Karpenko, Maksimova 2010] : A weak transitive algebra $\mathfrak{A}$ is simple iff for all x in $\mathfrak{A}$ $x\&\Box x = \boxdot x = \begin{cases} 1, & \text{if } x = 1; \\ 0, & \text{if } x \neq 1. \end{cases}$

Theorem [Karpenko, Maksimova 2010]: Every simple weak transitive algebra is *DL*-algebra.

V_n^m algebras

By V_n^m we denote a modal algebra with $(n+m)$ atoms $a_1, \ldots, a_n$, $b_1, \ldots, b_m$ such that, for every atom x:

$$\Diamond x = \begin{cases} 1, & x = a_i \quad \text{for some } 1 \leq i \leq n; \\ \neg x, & x = b_j \quad \text{for some } 1 \leq j \leq m; \end{cases}$$

Theorem [Karpenko, Maksimova 2010]: Every finitely generated finitely indecomposable DL-algebra is simple and isomorphic to the algebra V_n^m for the suitable $n + m > 0$.

System of embedding

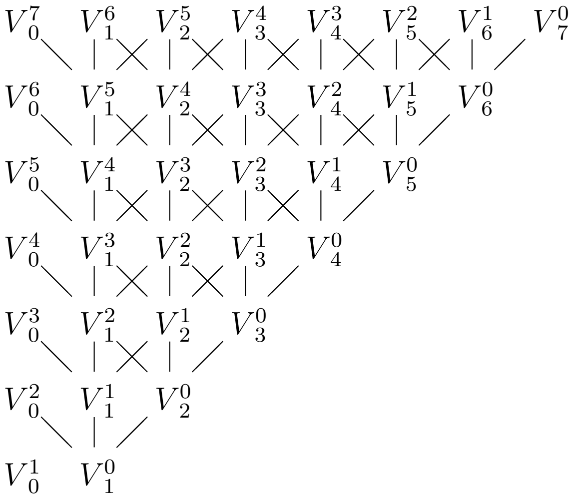
For an arbitrary finite DL-algebras $\mathfrak{A}$ and $\mathfrak{B}$, $\mathfrak{A} \preceq \mathfrak{B}$ denotes that $\mathfrak{A}$ is isomorphically embedded into $\mathfrak{B}$.

Theorem [Karpenko, Maksimova 2010]: The relation $\preceq$ is a reflexive and transitive closure of the relation

$$V_n^m \preceq V_{n+1}^m, V_n^m \preceq V_n^{m+1}, V_n^m \preceq V_{n-1}^{m+2},$$

for $m \geq 0, n \geq 1$.

System of embedding



One to one correspondence

By $K(DL)$ we denote a class of all algebras V_n^m where $n + m > 0$.

The class K is called *closed downward* if for any algebra $\mathfrak{A} \in K$ the following condition $(\mathfrak{B} \preceq \mathfrak{A} \Rightarrow \mathfrak{A} \in K)$ is valid.

Theorem [2010]: There is a one to one correspondence between varieties of DL-algebras and subclasses of $K(DL)$ closed downward.

Amalgamable varieties of *DL*-algebras

Theorem [2010]: There are exactly 16 amalgamable varieties of *DL*-algebras, the corresponding subclasses of $K(DL)$ are the following:

1. empty;
2. generated by V_0^1;
3. generated by V_1^0;
4. generated by V_1^0 and V_0^1;
5. generated by V_0^2;
6. generated by V_0^2 and V_0^1;

7. generated by V_1^1;
8. generated by V_1^1 and V_0^1;
9. generated by V_2^0;
10. generated by V_2^0 and V_0^1;
11. generated by V_0^3;
12. generated by V_0^3 and V_0^1;
13. generated by V_0^4;
14. generated by V_0^4 and V_0^1;
15. generated by all algebras V_n^0 $(n > 0)$;
16. generated by all algebras V_n^0 $(n > 0)$ and V_0^1.

Results 2010

Theorem [2010]: WIP and IPD are equivalent over DL.

Theorem [2010]: There are exactly 16 logics with IPD in $NE(DL)$.

Theorem [2010]: The logic $L \in NE(wK4)$ has WIP iff the logic $L + (A \to \Box\Diamond A)$ has WIP.

Characteristic formula of special type

For a given $x \in AtV_n^m = \{a_1, \ldots, a_n, b_1, \ldots, b_m\}$ we associate some variable p_x. The following formula $\theta(V_n^m)$ have the same properties with Characteristic formula:

$$\theta(V_n^m) = \Bigg(\Bigg(\Big(\mathop{\&}_{\substack{x,y \in AtV_n^m \\ x \neq y}} \boxdot \neg(p_x \& p_y)\Big) \& \boxdot \Big(\bigvee_{x \in AtV_n^m} p_x\Big) \& \\ \& \mathop{\&}_{x \in \{a_1,\ldots,a_n\}} \boxdot \Diamond p_x \& \mathop{\&}_{x \in \{b_1,\ldots,b_m\}} \boxdot (\Diamond p_x \leftrightarrow \neg p_x)\Big)\Bigg) \to 0\Bigg) \quad (1)$$

Axiomatization of extensions of DL with IPD

Theorem [2012]: Following extensions of *DL* and only they have *IPD*:

1. $(For)' = DL + \theta(V_1^0)$; $For = (For)' + \theta(V_0^1)$;
2. $(LV_1^0)' = DL + \theta(V_0^2) + \theta(V_1^1) + \theta(V_2^0)$;
   $LV_1^0 = (LV_1^0)' + \theta(V_0^1)$;
3. $(LV_0^2)' = DL + \theta(V_1^1) + \theta(V_2^0)$; $LV_0^2 = (LV_0^2)' + \theta(V_0^1)$;
4. $(LV_1^1)' = DL + \theta(V_2^0) + \theta(V_0^2) + \theta(V_0^3)$;
   $LV_1^1 = (LV_1^1)' + \theta(V_0^1)$;
5. $(LV_2^0)' = DL + \theta(V_3^0) + \theta(V_1^1) + \theta(V_0^2)$,
   $LV_2^0 = (LV_2^0)' + \theta(V_0^1)$;
6. $(LV_0^3)' = DL + \theta(V_0^2) + \theta(V_2^0)$; $LV_0^3 = (LV_0^3)' + \theta(V_0^1)$;
7. $(LV_0^4)' = DL + \theta(V_0^2) + \theta(V_0^3) + \theta(V_3^0) + \theta(V_2^1)$;
   $LV_0^4 = (LV_0^4)' + \theta(V_0^1)$;
8. $(S5)' = DL + \theta(V_0^2) + \theta(V_1^1)$; $S5 = (S5)' + \theta(V_0^1)$.

Results 2012

Theorem [2012]: The deductive interpolation property is decidable over *DL*.

Theorem [2012]: The amalgamation property is decidable for varieties of *DL*-algebras.

Theorem [2012]: The weak interpolation property is decidable over $wK4$.

From the paper of L. Maksimova [1980] follows that the logics $(LV_1^1)'$ and (LV_1^1) does not have the CIP.

The question about the number of extensions of *DL* with CIP is still open.

Next question

Does the class of finitely generated and simple n-transitive modal algebras have the amalgamation property?

Does the n-transitive modal logic have the WIP?

Simple n-transitive modal algebras

Theorem 1: A n-transitive algebra $\mathfrak{A}$ is simple iff for all x in $\mathfrak{A}$

$$[n]x = \begin{cases} 1, & \text{if } x = 1; \\ 0, & \text{if } x \neq 1. \end{cases}$$

Lemma 1: If the formula $[n]A \to \Box^{n+1}A$ is valid in a Kripke frame, then the frame satisfies the condition
$\forall u \forall v (uR^{n+1}v \Rightarrow (u = v \text{ or } \exists\ k : k \leq n \text{ and } uR^k v))$.

The frames which identify the condition of Lemma 1 we will call n-transitive.

Representation theorems

Let $\mathcal{W} =< W, R >$ be a given frame. Denote by $\mathcal{W}^+$ the power-set of $\mathcal{W}$ and let $\mathfrak{W}^+ = (W^+, \to, \bot, \Box)$ where $\bot = \emptyset, X \to Y = (W - X) \cup Y), \Box X = \{x \in |\forall y(xRy \Rightarrow y \in X)\}$ for all $X, Y \subseteq W$. Then $\mathfrak{W}^+$ is a modal algebra.

Let $\mathfrak{A} = (|\mathfrak{A}|, \to, \bot, \Box)$ be a modal algebra, $W(\mathfrak{A})$ the set of it ultrafilters, $R(\mathfrak{A})$ a relation on $\mathfrak{A} = (|\mathfrak{A}|, \to, \bot, \Box)$ defined as follows: $\Phi R(\mathfrak{A})\Psi$ iff $\forall x(\Box x \in \Phi \Rightarrow x \in \Psi)$.
Let $\mathfrak{W}(A) = (W(\mathfrak{A}), R(\mathfrak{A}))$.

Representation theorem of Jonsson and Tarski [1951]: For each modal algebra $\mathfrak{A}$, the mapping $\varphi(x) = \{\Phi \in W(\mathfrak{A}) | x \in \Phi\}$, is a monomorphism from $\mathfrak{A}$ into $\mathfrak{W}^+(A)$. Moreover, if A is finite, φ is an isomorphism onto $\mathfrak{W}^+(A)$.

For each finite modal algebra $\mathfrak{A}$ the frame $\mathfrak{W}(A)$ may be replaced by the frame $At(\mathfrak{A})$ of all atoms of $\mathfrak{A}$, where a relation R is defined by $aRb \Leftrightarrow a \leq \Diamond b$ for all a, b in $At(\mathfrak{A})$.

Following [Karpenko, Maksimova, 2010] we denote the frame $\mathcal{X}_n^m =< X_n^m, R >$, where $X_n^m = \{a_1, \dots, a_n, b_1, \dots, b_m\}$ and

$$(uRv \Leftrightarrow (u \neq v \text{ or } u = a_i \text{ for some } 1 \leq i \leq n))$$

which is isomorphic to the frame of atoms $At(V_n^m)$.

For a given finite algebras $\mathfrak{A}_1$ and $\mathfrak{A}_2$ the condition $\mathfrak{A}_1 \preceq \mathfrak{A}_2$ is equivalent for the existence of p-morphism from $At(\mathfrak{A}_2)$ onto $At(\mathfrak{A}_1)$.

Results

Lemma 2: Let F be a frame and $\theta_1 : F \to \mathcal{X}_0^2$ and $\theta_2 : F \to \mathcal{X}_0^3$ are p-morphisms. Then the frame F is not 2-transitive.

Corollary 2: The class of finitely generated, finite and simple 2-transitive modal algebras dose not have AP.

Results

By CV^n we denote a modal algebra with atoms $b_1, \ldots, b_n$ such that $\Diamond b_n = b_1, \Diamond b_i = b_{i+1}$ for $1 \leq i \leq n-1$.

By the Lemma 1 CV^n is simple n-transitive modal algebra.

By $\mathcal{CV}^n$ we denotes the correspondence frame $< At(CV^n), R(CV^n) >$.

Results

Lemma 3: Let F be a frame and $\theta : F \to \mathcal{CV}^n$ be a mapping. Then θ is a p-morphism iff the following condition holds:

1. $\forall x, y \in \theta^{-1}(c_i)$ dose not hold (xRy) for all $i \in \{1, \dots, n\}$;
2. $\forall x \in \theta^{-1}(c_i) \exists y \in \theta^{-1}(c_{i+1})$ such that xRy for all $\forall i \in \{1, \dots, n-1\}$;
3. $\forall x \in \theta^{-1}(c_n) \exists y \in \theta^{-1}(c_1)$ such that xRy;
4. $\forall x \in \theta^{-1}(c_i) \forall y \in \theta^{-1}(c_k)(\neg c_i R c_k \Rightarrow \neg xRy)$.

Lemma 4: Let F be a frame and $\theta_1 : F \to \mathcal{CV}^n$ and $\theta_2 : F \to \mathcal{CV}^{n-1}$ are p-morphisms. Then the frame F is not n-transitive (n $\geq$ 3).

Theorem: The class of finitely generated, finite and simple n-transitive modal algebras does not have AP.

Thank you for your attention.