Problems of Unification and Admissible Rules in Non-Classical Logics (with applications to AI, CS and CD)

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History, Pre-History

(1955) Lorentzen: Clear definition of admissible rules.

(1975) Harvey Fridman problem: if IPC H is decidable w.r.t admissible rules.

Solution: yes – Rybakov, 1984 - IPC, $S4, Grz$ + etc.

(1975) A.Kuztetsov problem: if IPC H has a finite basis for inference rules: -

Solution: no – Rybakov 1985-86.

(1950???) P.Novikov Problem: decidability logical equations in IPC + etc.

Solution: yes - Rybakov -1986

(1999) S.Ghilardi - decidability of admissibility in IPC via unification and projective formulas (projective approximation) - new solution for Friedman problem. Silvio Ghilardi: Unification in Intuitionistic Logic. J. Symb. Log. 64(2): 859-880 (1999)

(2004) S.Ghilardi - unification via extension property in general modal frames, admissibility in modal logics S4, S4.2.

(2001) R. Iemhoff. Explicit basis for rules admissible in IPC - On the admissible rules of intuitionistic propositional logic. Journal of Symbolic Logic 66, 2001 (p. 281-294).

(2001) V. Rybakov. Explicit basis for rules admissible in modal logic $S4$. : Construction of an Explicit Basis for Rules Admissible in Modal System S4. Math. Log. Q. 47(4): 441-446 (2001).

(2009) R. Iemhoff and G. Metcalfe. Proof theory for admissible rules. Annals of Pure and Applied Logic 159 (1-2), 2009 (p.171-186).

Short recall of definitions and notation

Definition. An inference rule $\varphi_1, \ldots, \varphi_n/\psi$ is said to be *admissible* in a logic L if for any substitution ε the following holds: if $\varepsilon(\varphi_1) \in L, \ldots, \varepsilon(\varphi_n) \in L$ then $\varepsilon(\psi) \in L$,

Definition. A formula φ is unifiable in a logic L if there is a substitution ε (which is called a unifier for φ) such that $\varepsilon(\varphi) \in L$.

Definition. *A unifier ε (for a formula φ in a logic L) is more general than a unifier ε_1 iff there is a substitution δ such that for any letter x, $[\varepsilon_1(x) \equiv \delta(\varepsilon(x))] \in L$.*

If a logic L is decidable, to check the unifiability a formula in L is (theoretically, not computationally) an easy task: it is sufficient to use only ground substitutions: mappings of variable-letters in the set $\{\perp, \top\}$. But the problem - how to find all unifiers - all solving substitutions - is not easy at all.

Definition. A set of unifiers CU for a given formula φ in a logic L is a complete set of unifiers, if the following holds. For any unifier σ for φ in L, there is a unifier σ_1 from CU, where σ_1 is more general than σ.

Definition. A logic L has unitary unification if any unifiable in L formulas has a complete set of unifiers consisting of single formula (we call it mgu - most general unifier)

If a logic L has finite computable set of unifiers for any unifiable formula, L is decidable w,r.t. admissible rules: it is sufficient to verify only these unifiers for checking admissibility.

For LTL (which possesses definable $\Box$ and $\Diamond$,
- $\Diamond x := (\top \mathbf{U} x)$, $\quad \Box = \neg\Diamond\neg$
or modal logics over $S4$ we may formulate projectivity as follows:

Definition. A formula φ is said to be **projective** in a logic L if the following holds. There is a substitution σ (which is called projective substitution, projective unifier) such that σ is a unifier and $\Box\varphi \to [x_i \equiv \sigma(x_i)] \in L$ for any letter x_i from φ.

Proposition *If a substitution σ_p is projective for a formula φ in a logic L, then $\{\sigma_p\}$ is a complete set of unifiers for φ (i.e. σ_p is most general unifier).*

Proof. Indeed, let σ be a unifier for φ in L. Since we assume σ_p is projective for φ in L, we have $\Box\varphi \to [x_i \equiv \sigma_p(x_i)] \in L$ for any letter x_i from φ. Acting by σ on the formula above we get $\sigma(\Box\varphi) \to [\sigma(x_i) \equiv \sigma(\sigma_p(x_i))] \in L$, that is $\sigma(x_i) \equiv \sigma(\sigma_p(x_i)) \in L$. Q.E.D.

L has projective unification if every unifiable formula has a projective unifier.

Projectivity implies unitary unification, but not vise versa:

Modal logic $S4.2$ has unitary unification (Ghilardi, Sacchetti, 2006), but

Example. The unifiable formula

$$\Box(\Box x \to \Box y) \to \Box x \vee \Box z$$

has an mgu in $S4.2$ but can not have a projective unifier in $S4.2$.

Survey: Recent Solutions of open questions in 'standard' areas ...

Emil Jerábek. Complexity of admissible rules, Archive for Mathematical Logic 46 (2007):

admissibility in typical normal extensions of K4 (K4, GL, S4, S4Grz)and s.i. logics (IPC +++) is coNEXP-complete (and in particular, strictly more complex than the drivability problem, under reasonable complexity-theoretic assumptions).

Emil Jerábek. Independent bases of admissible rules, Logic Journal of the IGPL 16 (2008).

IPC, K4, GL, and S4, as well as all logics inheriting their admissible rules, have independent bases of admissible rules.

A move for new open problems, to new horizon

(i) Lukasiewicz logic:

Emil Jerábek. Admissible rules of Lukasiewicz logic, Journal of Logic and Computation 20 (2010)

Lukasiewicz multi-valued propositional logic: admissibility of multiple-conclusion rules in Lukasiewicz logic, as well as validity of universal sentences in free MV-algebras, is decidable (in PSPACE).

Emil Jerábek. Bases of admissible rules of Lukasiewicz logic, Journal of Logic and Computation 20 (2010)

Explicit bases of single-conclusion and multiple-conclusion admissible rules of propositional Lukasiewicz logic, also – a proof that Lukasiewicz logic has no finite basis of admissible rules.

Linear temporal logic LTL with UNTIL and NEXT

(2008)V. Rybakov. Linear temporal logic LTL with until and next, logical consecutions. Annals of Pure and Applied Logic 155, 2008.

LTL is decidable w.r.t admissible inference rules. As a consequence we obtain algorithms verifying the validity quasi-identities in varieties of corresponding algebras.

(2011) S. Babenyshev, V. Rybakov. Linear temporal logic LTL: basis for admissible rules Journal of Logic and Computation, 2011.

Provide an explicit (infinite) basis for rules admissible in LTL.

(2012) V. V Rybakov. Writing out Unifiers in Linear Temporal Logic Journal of Logic and Computation 22, 2012.

Any unifiable in LTL formula has a most general unifier (thus, LTL enjoys unitary unification). The algorithm of construction such MGU is provided. This solves unifiability problem for LTL and the admissibility problem.

Unification with Coefficients

V. Rybakov. Writing out unifiers for formulas with coefficients in intuitionistic logic Logic Journal of IGPL, 2013.

V. Rybakov. Unifiers in transitive modal logics for formulas with coefficients (meta-variables) Logic Journal of IGPL, 2013.

Solution of the unification problem in these logics for formulas with coefficients (meta-variables).

LTL **with** $SINCE$ **and similar (e.g. simply temporal logics with nodes) - EASY via modeling universal modality:**

2008, V Rybakov. Multi-modal and temporal logics with universal formulareduction of admissibility to validity and unification. Journal of logic and computation, 2008.

Paraconsistent minimal Johanssons' logic J and positive intuitionistic logic

2013, S. Odintsov, V. Rybakov. Unification and admissible rules for paraconsistent minimal Johanssons' logic J and positive intuitionistic logic IPC+ . Annals of Pure and Applied Logic, 2013.

This paper proves that the problem of admissibility for inference rules with coefficients (parameters)(as well as plain oneswithout parameters) is decidable for the paraconsistent minimal Johanssons' logic J and the positive intuitionistic logic IPC+. Using obtained technique we show also that the unification problem for these logics is also decidable: we offer algorithms which compute finite complete sets of unifiers for any given unifiable formula.

Description logics

F. Baader, S. Ghilardi Unification in Modal and Description Logics, Logic Journal of the IGPL, vol.19, n.6, pp. 705-730, 2011.

via Algebra

S. Ghilardi. Unification, Finite Duality and Projectivity in Locally Finite Varieties of Heyting Algebras , Annals of Pure and Applied Logic, vol. 127/1-3, pp.99-115 (2004).

APPLICATIONS to AI, CS and CD

V.Rybakov. Algorithm for Decision Procedure in Temporal Logic Treating Uncertainty, Plausibility, Knowledge and Interacting Agents, International Journal of Intelligent Information Technologies (IJIIT) 6 (2010).

Logic UIA_{LTL}, which is a combination of the linear temporal logic LTL, a multi-agent logic with operation for passing knowledge via agents' interaction, and a suggested logic based on operation of logical uncertainty. The logical operations of UIA_{LTL} also include (together with operations from LTL)

- operations of strong and weak until,

- UIA_{LTL} agents' knowledge operations,

- UIA_{LTL}operation of knowledge via interaction,

- UIA_{LTL} operation of logical uncertainty,

- UIA_{LTL} the operations for environmental and global knowledge.

V Rybakov. Interpretation of chance discovery in temporal logic, admissible inference rules. - In: Knowledge-Based and Intelligent Information and Engineering Systems, KES-2010. LNCS,2010.

CD - in terms of plausibility to discover in search.

V Rybakov, S Babenyshev. Multi-agent logic with distances based on linear temporal frames Artificial Intelligence and Soft Computing, 337-344, 2010.

Distance - from k to $k+m$ steps - possible to discover in this distance.

V. Rybakov. Representation of knowledge and uncertainty in temporal logic LTL with since on frames Z of integer numbers. Knowledge-Based and Intelligent Information and Engineering Systems, 306-315, 2011, Springer, LNCS.

Uncertainty via combination of evidences in future and past.

Projectivity in linear temporal logics LTL with UNTIL

(Origin) Linear Temporal Logic LTL with Next and Until:

Amir Pnueli, 1977: LTL was first proposed for the formal verification of computer programs.

Manna and Pnueli: The temporal logic of Concurrent and Reactive Systems, 1992.

Moshe Y. Vardi. An Automata-Theoretic Approach to Linear Temporal Logic, since 1995

Short recall of definitions:

LTL is built up from a finite set of propositional variables AP, the logical operations $\neg$ and $\vee$, and the temporal modal operations $\mathbf{N}$ (next time) and $\mathbf{U}$ (until). Formally, the set of LTL formulas over AP is inductively defined as follows:

If $p \in AP$ then p is a LTL-formula;

If ψ and φ are LTL-formulas then

$\neg\psi$, $\psi \vee \varphi$, $\mathbf{N}\varphi$ and $\psi\mathbf{U}\varphi$, are LTL formulas.

Semantics for **LTL** consists of runs of computation with given evaluations of propositional variables AP. Formally they may be viewed as Kripke models with base sets to be natural numbers, with standard understanding meaning of next (interpretation of $\mathbf{N}$), and with a given valuation V of AP. $M := \langle N, \mathbf{N}, V \rangle, \quad \forall p \in AP, V(p) \subseteq N.$ Formally, the satisfaction (truth) relation between a word and an LTL formula is defined as follows: $\forall w \in N$,

$$w \models_V p \iff p \in V(p);$$

$$w \models_V \neg\varphi \iff not(w \models_V \varphi);$$

$$w \models_V \varphi \vee \psi \iff w \models_V \varphi \ \ or \ \ w \models_V \psi$$

$$w \models_V \mathbf{N}\varphi \iff (w+1) \models_V \varphi;$$

$$w \models_V \varphi \mathbf{U} \psi \iff \exists k \in N[(w+k) \models_V \psi \ and \ \forall n < k (w+n) \models_V \varphi].$$

(φ must remain true until ψ becomes true)

Linear modal logic $S4.3$ since (long ago (1983)) was known to be decidable about admissibility and structure of admissibility bases. But recently an algorithm for constructing projective unifiers in logics extending $S4.3$ was offered in

W.Dzik and P. Wojtylak (2011); Wojciech Dzik, Piotr Wojtylak: Projective unification in modal logic. Logic Journal of the IGPL 20(1): 121-153 (2012).

by a technique using Löwenheim substitutions and CNF-forms in such logics. So, any unifiable formula there is projective and hence has a computable mgu.

With linear temporal logic LTL the case is more complicated:

Proposition (Rybakov, 2012) *Formula* $\varphi = \Box(\Box x \vee (\neg x \wedge \mathbf{N}\Box x))$ *is unifiable in* $\mathbf{LTL}$ *but not projective.*

Proof. Substitution $x \mapsto \top$ is an obvious unifier for φ. Suppose now φ is projective and π is a corresponding projective unifier. Consider the run N_V (starting from 0: $|N_V| := \{0, 1, 2, \ldots\}$):

$$\overset{x}{\circ} \xrightarrow{N} \overset{\neg x}{\circ} \xrightarrow{N} \overset{\Box x}{\circ} \xrightarrow{N} \overset{\Box x}{\circ} \;\ldots$$

Since $(N_V, 1) \Vdash_V \Box\varphi$, then $(N_V, 1) \Vdash_V x \leftrightarrow \pi(x)$. Therefore, notwithstanding either $(N_V, 0) \Vdash_V \pi(x)$ or $(N_V, 0) \Vdash_V \neg\pi(x)$, we have that $(N_V, 0) \Vdash_V \neg\Box\pi(x)$ and, at the same time, $(N_V, 0) \Vdash_V \neg\mathbf{N}\Box\pi(x)$. Thus $(N_V, 0) \Vdash_V \neg\pi(\varphi)$, hence π cannot be an φ-unifier, a contradiction. Q.E.D.

Bearing in mind our task to push anyway projectivity to $\mathbf{LTL}$, we will consider $\mathbf{LTL_U}$ - the fragment of LTL with the operation $\mathbf{U}$ but without next - $\mathbf{N}$ (that is formulas of this fragment do not contain $\mathbf{N}$).

Since basic operation - until - $\mathbf{U}$ is presented in $\mathbf{LTL_U}$, we can again define basic modal operations - $\Box$ and $\Diamond$ and define projectivity as earlier.

Theorem Any unifiable in $\mathbf{LTL_U}$ formula φ is projective.

Take any $X \subseteq Sub(\Box\varphi)$, let

$$\Psi(X) := \Box\varphi \wedge \bigwedge_{\psi U\xi \in X} \psi U\xi \quad \wedge \bigwedge_{\psi,\xi \in Sub(\Box\varphi), \psi U\xi \notin X} \neg(\psi U\xi).$$

$$\sigma(x_i) := (\Box\varphi(x_1,...,x_n) \wedge x_i) \ \vee (\neg\Box\varphi \wedge \Diamond\Box\varphi$$

$$\wedge \bigvee_{\Psi(X)\in Sat} \Box[\neg\Box\varphi \wedge \Diamond\Box\varphi \rightarrow \neg\Box\varphi \mathbf{U} \Psi(X) \wedge T(\Psi(X), x_i))]) \vee$$

$$(\neg\Diamond\Box\varphi \wedge T(x_i))\,.$$

Thus, $\mathbf{LTL_U}$ enjoys projective unification and any unifiable formula has computable mgu. This solves open problem of recognizing rules admissible in LTL.