Products of modal logics

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- In the early days of modal logic (before 1980s) there was interest in studying multiple particular systems.

Contemporary modal logic also investigates classes of logics and general constructions combining different systems.

- Products were introduced in the 1970s; their intensive study started in the 1990s.

Motivations for studying products of modal propositional logics

- A natural type of combined modal logics
- Connection to first-order classical logic
- Connection to first-order modal logics
- Connection to relation algebras
- Connection to description logics

The main reference for products (BOOK03)

D. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev. Many–dimensional Modal Logics: Theory and Applications. Elsevier, 2003.

PRODUCTS OF FRAMES

Kripke n-frames: $(W,R_1,\ldots,R_n)$ (relational system with n binary relations).

Def. The product of two Kripke frames

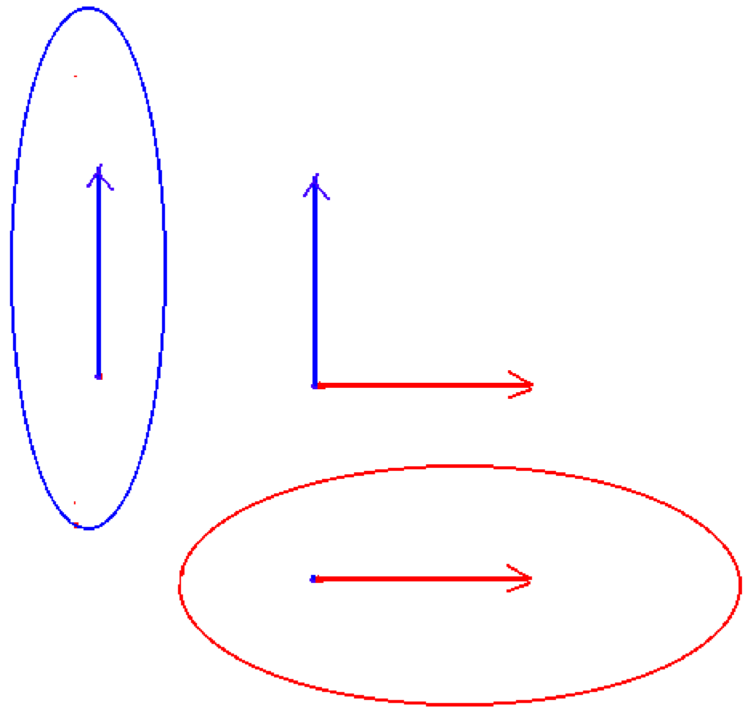
$$(W,R_1,\ldots,R_n) \times (V,S_1,\ldots,S_m) := (W\times V,R_{11},\ldots,R_{n1},S_{12},\ldots,S_{m2})$$

where

$$(x_1,y_1)R_{i1}(x_2,y_2) \Leftrightarrow x_1R_ix_2 \ \&\ y_1=y_2$$

$$(x_1,y_1)S_{j2}(x_2,y_2) \Leftrightarrow x_1=x_2 \ \&\ y_1S_jy_2$$

Multiple products $F_1 \times\ldots\times F_n$ are defined in an obvious way. The multiplication is associative up to isomorphism.



PRODUCTS OF MODAL LOGICS

Normal n-modal logics are defined us usual - as sets of modal formulas in the propositional language with unary modal connectives $\Box_1,\dots,\Box_n$ containing the minimal logic and closed under standard rules.

Every Kripke frame is associated with a modal logic – the set of all valid formulas:

$$\mathbf{L}(F) := \{A \mid F \models A\}.$$

Logics of this form are called (Kripke) complete.

If F is finite, **L**(F) is called tabular.

For a class of frames $\mathcal{C}$

$$\mathbf{L}(\mathcal{C}) := \cap\{\mathbf{L}(F) \mid F \in \mathcal{C}\}.$$

If all frames in $\mathcal{C}$ are finite, $\mathbf{L}(\mathcal{C})$ has the finite model property (fmp).

A modal logic L defines the class of L-frames

$$\mathbf{V}(L) := \{F \mid F \models L\}.$$

L is called elementary if $\mathbf{V}(L)$ is an elementary (first-order definable) class in the classical sense.

Remark L is complete iff L=**L**(**V**(L)).

Some particular complete logics

$\mathbf{K}_n$ is the minimal n-modal logic, $\mathbf{K}=\mathbf{K}_1$.
$\mathbf{K.t}_n$ is the minimal n-temporal logic, $\mathbf{K.t}=\mathbf{K.t}_1$.
$\mathbf{K.t}_n$ -frames are $(W,R_1,(R_1)^{-1},\ldots,R_n,(R_n)^{-1})$.
$\mathbf{T} = \mathbf{K}+\Box p\to p = \mathbf{L}$(all reflexive frames)
$\mathbf{K4} = \mathbf{K}+\Box p\to\Box\Box p = \mathbf{L}$(all transitive frames)
$\mathbf{S4} = \mathbf{K4}+\Box p\to p = \mathbf{L}$(all transitive reflexive frames).
$\mathbf{K4.3} = \mathbf{L}$(all transitive non-branching frames)
$=\mathbf{L}$(all strict linear orders)
$\mathbf{S4.3} = \mathbf{K4.3}+\Box p\to p = \mathbf{L}$(all linear orders)
$\mathbf{S5} = \mathbf{S4}+\Diamond\Box p\to p = \mathbf{L}$(all equivalence frames)
$=\mathbf{L}$(all universal frames)
$\mathbf{Grz} = \mathbf{L}$(all finite posets)
$\mathbf{GL} = \mathbf{L}$(all strict finite posets)
$\mathbf{Grz3} = \mathbf{L}$(all finite chains)
$\mathbf{GL3} = \mathbf{L}$(all strict finite chains)

Def. The product of two modal logics

$$L_1 \times L_2 := \mathbf{L}(\{F_1 \times F_2 \mid F_1 \models L_1 , F_2 \models L_2 \}.$$

Similarly we can define multiple products

$$L_1 \times \ldots \times L_n := \mathbf{L}(\{F_1 \times \ldots \times F_n \mid F_1 \models L_1 ,\ldots, F_n \models L_n \}.$$

However, multiplication of logics is probably non-associative (an open problem).

AXIOMATIZATION: FIRST RESULTS

AXIOMATIZATION PROBLEM: *to find axioms of* $L_1 \times \ldots \times L_n$ *given the axioms of* $L_1,\ldots, L_n$.

Theorem 1 (Sh 1987, Gabbay&Sh 1998)

Classes of frames $\mathcal{C}_1,\ldots, \mathcal{C}_n$ are elementary $\Rightarrow \mathbf{L}(\mathcal{C}_1 \times \ldots \times \mathcal{C}_n)$ is RE.

(So, $L_1,\ldots, L_n$ are Kripke complete and elementary $\Rightarrow L_1 \times \ldots \times L_n$ is RE.

Corollary 1.1 (Sh 1987) $\mathbf{L}((\mathbf{Q},<)^2), \mathbf{L}((\mathbf{Q}, \leq)^2)$ are RE.

Def. The fusion of two modal logics with disjoint modalities

$L_1 * L_2$:= the smallest logic containing $L_1 \cup L_2$

Remarks on fusions

Fusion of logics preserves many properties:

Theorem 2 (Kracht&Wolter 1991, Fine&Schurz 1996) Fusion preserves Kripke completeness, the fmp, decidability.

Bad news: products do not preserve any of these properties.
Good news: sometimes they still do.

Def. The commutative join of two modal logics with disjoint modalities

$\Box_i$ $(1 \leq i \leq n)$, $\blacksquare_k$ $(1 \leq j \leq m)$

is

$[L_1, L_2] := L_1 * L_2 +$

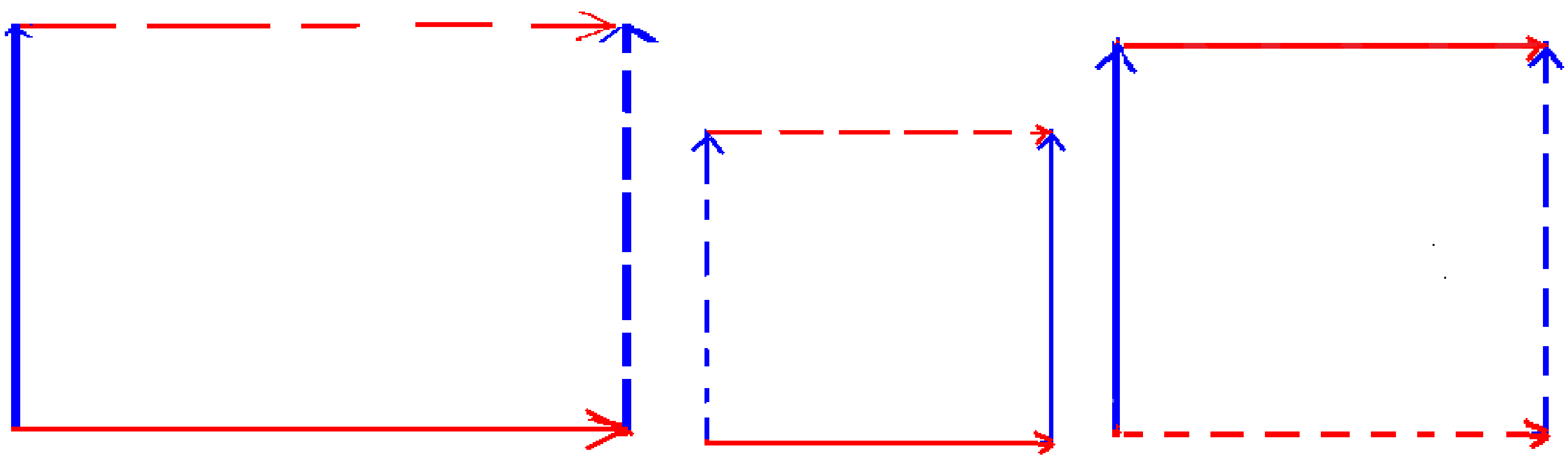
$\Diamond_i \blacksquare_k p \to \blacksquare_k \Diamond_i p + \Box_i \blacksquare_k p \leftrightarrow \blacksquare_k \Box_i p$ for any i, k

Remark. If the modalities are not disjoint, we rename them.

The additional axioms are Sahlqvist formulas expressing the following properties of the relations in the product frame

$(R_{i1})^{-1} \circ S_{k2} \subseteq S_{k2} \circ (R_{i1})^{-1}$ (Church - Rosser property)

$R_{i1} \circ S_{k2} = S_{k2} \circ R_{i1}$ (commutativity)



Def. L_1, L_2 are product matching if $[L_1, L_2] = L_1 \times L_2$

Def. A Horn sentence is a universal first order sentence of the form

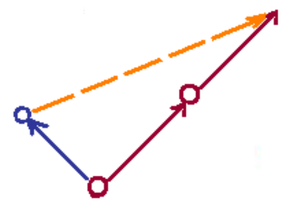
$$\forall x \ldots (\varphi(x,y,\mathbf{z}) \to R(x,y)),$$

where φ is positive, $R(x,y)$ is atomic.

A modal formula A is Horn if it corresponds to a Horn sentence (i.e., the class of its frames **V**(A) is definable by a Horn sentence).

Example Modal formulas of the form $(\Diamond \ldots \Diamond)\Box p \to (\Box \ldots \Box)p$

correspond to Horn sentences



Logics with such axioms are always complete.

Def. A modal logic is Horn axiomatizable if it is axiomatizable by formulas that are either variable-free or Horn.

Completeness theorem for products

([Gabbay, Sh 1998]<< [BOOK03])

Theorem 3 If L_1, L_2 are Kripke complete and Horn axiomatizable, then they are product matching.

Counterexamples

Theorem 4 [Sh 1987 << Gabbay,Sh 1998]

Let **L** be a nontrivial 1-modal logic containing **Grz**. Then **L** and **S5** are not product-matching.

Theorem 5 [Kurucz & Marcelino 2011]

K4.3 and **S5, S4.3** and **S5** are not product-matching.

Storнger counterexamples: finite axiomatizability is not preserved

(see later)

FMP AND PRODUCT FMP

Def. A QTC-logic is axiomatizable by variable-free formulas and

formulas or axioms of the form $\Diamond_i \Box_j p \to p$, $\Box_i p \to (\Box_i)^k p$.

Theorem 6 [Sh 2005] If L_2 is a QTC-logic, then

$\mathbf{K.t}_n \times L_2 = [\mathbf{K.t}_n, L_2]$ has the fmp.

Theorem 7 [Sh 2011] $(\mathbf{K.t}_n)^2 = [\mathbf{K.t}_n, \mathbf{K.t}_n]$ has the product fmp.

THE LACK OF RECURSIVE AXIOMATIZATION

(Reynolds&Zakharyaschev 2001<< BOOK03)

Theorem 8 (a) If $\mathcal{C}_1$, $\mathcal{C}_2$ are classes of **K4.3**-frames containing

some frames with descending ω-chains and every frame in is Dedekind-complete (i.e., every bounded set has supremum), then $\mathbf{L}(\mathcal{C}_1 \times \mathcal{C}_2)$ is Π^1_1-hard.

(b) If $\mathcal{C}_1$, $\mathcal{C}_2$ are classes of **K4.3**-frames containing some frames

with ascending ω-chains and every frame in $\mathcal{C}_1$ is Dedekind-

complete, then $\mathbf{L}(\mathcal{C}_1 \times \mathcal{C}_2)$ is Π^1_1-hard.

Corollary 8.1 $\mathbf{GL3}^2$, $\mathbf{Grz3}^2$, **GL3**×**Grz3** are Π^1_1-hard.

Thus products do not preserve any interesting property of modal logics.

Corollary 8.2 **L**(F × G) is Π^1_1-hard whenever F is **R** or ω, F is **Q**, **R** or ω (with < or ≤).

Theorem 9 If $\mathcal{C}_1$, $\mathcal{C}_2$ are classes of finite [strict] linear orders of unbounded length, then $\mathbf{L}(\mathcal{C}_1 \times \mathcal{C}_2)$ is Π^0_1-complete.

Corollary 9.1 **L**($(\omega,>)^2$), **L**($(\omega,\geq)^2$) are Π^0_1-complete.

PRODUCTS WITH TABULAR LOGICS

Theorem 10 (Sh 2013) (1) If L_1 has the fmp and L_2 is tabular, then $L_1 \times L_2$ has the product fmp.

(2) If L_1 is decidable, L_2 is tabular, then $L_1 \times L_2$ is decidable.

TRANSLATION INTO MODAL PREDICATE LOGIC

Consider n-modal predicate formulas with arbitrary predicates, but without equality, constants and function symbols.

Kripke frame semantics with constant domains

Propositional Kripke frames: $F = (W, R_1, \ldots, R_n)$

Predicate Kripke frames with constant domains:

$\Phi = (F, D)$, where D is nonempty.

F is the frame of worlds of Φ, D is the set of individuals.

Kripke models over Φ:

$M = (\Phi, V)$, where V is a valuation:

$V(P) \subseteq D^n \times W$ for every n-ary predicate letter P,

For every formula $A(x_1, \ldots, x_n)$ and $d_i \in D$ we construct a D-sentence $A(d_1, \ldots, d_n)$

Forcing relation $M, u \models B$ between $u \in W$ and a D-sentence B is defined by induction, in particular:

- $M, u \models P(d_1, \ldots, d_n)$ iff $(d_1, \ldots, d_n, u) \in V(P)$

- $M,u \vDash \Box_i B$ iff $\forall v \in R_i(u)\ M,v \vDash B$
- $M,u \vDash \forall x B$ iff $\forall d \in D\ M,u \vDash [d/x]B$

Def $M \vDash A(x_1,\ldots,x_n)$ iff $\forall u \in W\ M,u \vDash \forall x_1 \ldots \forall x_n A(x_1,\ldots,x_n)$

(validity in a frame) $\Phi \vDash A$ iff for any M over Φ, $M \vDash A$

$\mathbf{L}(\Phi) := \{A \mid \Phi \vDash A\}$ is the modal predicate logic of Φ.

Wajsberg-type translation

Wajsberg's translation interprets **S5** in classical first-order logic.
Similarly,
every propositional (n+1)-formula A (with modalities $\Box_1,\ldots,\Box_n,[\forall]$)
is translated into
a first-order n-modal formula $A^{\#}(y)$ with (maybe) a parameter y:

Every $q \in PV$ is associated with a unary predicate letter Q. Then

$q^{\#}(y) := Q(y)$ (for $q \in PV$)

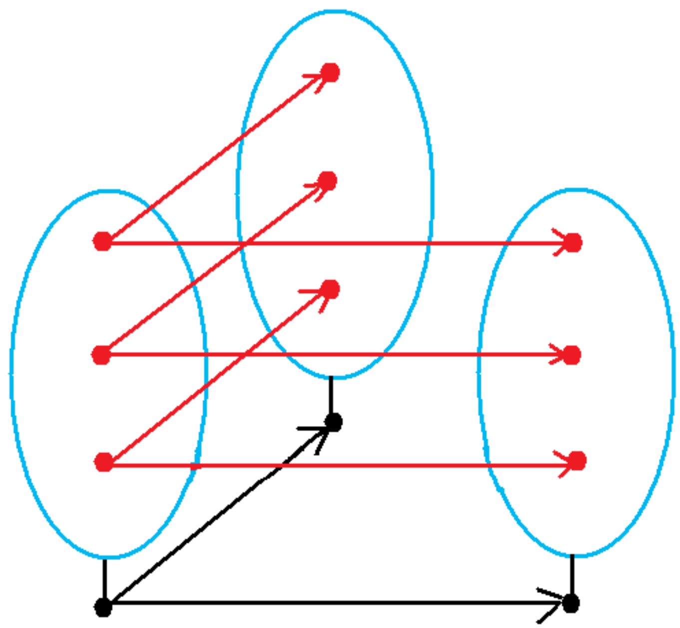
$\bot^{\#}(y) := \bot$

$(A \to B)^{\#}(y) := A^{\#}(y) \to B^{\#}(y)$

$(\Box_i A)^{\#}(y) := \Box_i A^{\#}(y)$

$([\forall]A)^{\#}(y) := \forall y A^{\#}(y)$

Lemma $F \times (D, D \times D) \models A$ iff $(F, D) \models \forall y A^{\#}(y)$



Theorem 11 Let L_1 be an n-modal propositional logic, $CK(L_1)$ the class of all predicate Kripke frames (F,D), with $F \vDash L_1$.
Consider the corresponding predicate modal logic $\mathbf{L}(CK(L_1))$. Then $L_1\times\mathbf{S5}$ is (polynomially) reducible to $\mathbf{L}(CK(L_1))$:
for any (n+1)-modal propositional A

$$L_1\times\mathbf{S5} \vdash A \text{ iff } \forall y A^{\#}(y)\in\mathbf{L}(CK(L_1))$$

(In other words, $L_1\times\mathbf{S5}$ specifies a fragment within $\mathbf{L}(CK(L_1))$

Completeness theorems for modal predicate logics yield a standard axiomatization of $\mathbf{L}(CK(L_1))$ in some cases.
QL_1 is the pure quantified version of L_1, BF is the conjunction of Barcan schema for all modalities:

$$\forall x \Box_i A \rightarrow \Box_i \forall x A.$$

Theorem 12 (1) (Tanaka&Ono, 1999) If L_1 is complete and $\mathbf{V}(L_1)$ is universally axiomatizable (in the classical sense), then

$$\mathbf{L}(CK(L_1)) = QL_1 + BF$$

(2) (Ono, 1983<<Gabbay&Skvorstov&Shehtman, 2009)

The same holds if L_1 is tabular.

In all these cases $L_1 \times \mathbf{S5}$ is RE, but now from Theorems 11, 12, 3 we obtain

Corollary 12.1 $[L_1, \mathbf{S5}] \vdash A$ iff $QL_1 + BF \vdash \forall y A^{\#}(y)$

whenever L_1 is Horn axiomatizable.

TRANSLATION INTO CLASSICAL PREDICATE LOGIC

This "square translation" resembles the well-known standard translation of modal formulas in the language of $(\mathbf{K}_n)^2$ into classical first-order formulas with relativized quantifiers. Consider the first order language with binary predicate letters $R_1,\dots R_n, P_1, P_2\dots$ We associate a binary predicate letter P_i with every proposition letter p_i.

$$(p_i)^2(x,y) := P_i(x,y)$$

$$(A \to B)^2(x,y) := A^2(x,y) \to B^2(x,y)$$

$$\bot^2(x,y) := \bot$$

$$(\Box_i A)^2(x,y) := \forall z\ (R_i(x,z) \to A^2(z,y))$$

$$(\blacksquare_i A)^2(x,y) := \forall z\ (R_i(y,z) \to A^2(x,z))$$

Theorem 13 If L is an elementary modal logic, φ (first-order) axiomatizes **V**(L), then

(1) QCL + φ $\vdash \forall x \forall y A^2(x,y)$ iff $L^2 \vdash A$.

(Here QCL is the classical first-order theory axiomatized by φ).

(2) If L^2 has the product fmp, then the corresponding "square fragment" of QCL + φ with binary predicates has the fmp (in the classical sense).

Axiomatizing some products of non-product-matching logics

Def. A propositional modal logic L is called locally tabular if, up to equivalence in L, for any m there are finitely many formulas in m propositional variables.

It is well-known that every locally tabular logic has the fmp.

Def. A propositional 1-modal logic L above **K4** is of finite depth < m if all L-frames are of depth <m.

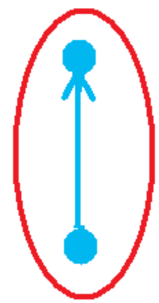
L is of depth m if it is of depth <m+1, but not <m.

Theorem 14(Segerberg, 1971) Every logic of finite depth is locally tabular.

Theorem 15 (Maksimova, 1974) The converse holds for extension of **K4**

Theorem 16 (Sh 2010) If L is of finite depth, then [L,**S5**] is locally tabular.

This allows us to axiomatize products of finite depth logics above **Grz** with **S5** in two ultimate cases: the catkin formula ACk is exactly what is missing. ACk is the Fine – Jankov formula of the following 2-frame (catkin):



Theorem 17 (Sh 2010) If

L=**Grz**+Adepth$_n$ (= **L**(all posets of depth n))

or

L=**Grz3**+ Adepth$_n$ (= **L**(all chains of depth n)),

then $\mathrm{L} \times \mathbf{S5} = [\mathrm{L}, \mathbf{S5}] + \mathrm{ACk}$.

Corollary 17.1 These logics are decidable.

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References

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[GKWZ03] D. Gabbay, A. Kurucz, F. Wolter, M. Zakharyaschev. Many-dimensional Modal Logics: Theory and Applications. Elsevier, 2003.

[Sano10] K. Sano. Axiomatizing hybrid products: How can we reason many-dimensionally in hybrid logic? J. Applied Logic, v.8, 2010, pp. 459-474.

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Theorem [Kurucz & Marcelino 2011]

K4.3×K, S4.3×K are not even axiomatizable in finitely many variables

QUESTION. Are the logics

K4.3 × S5, S4.3 × S5

finitely axiomatizable?

Squares of modal logics with additional connectives

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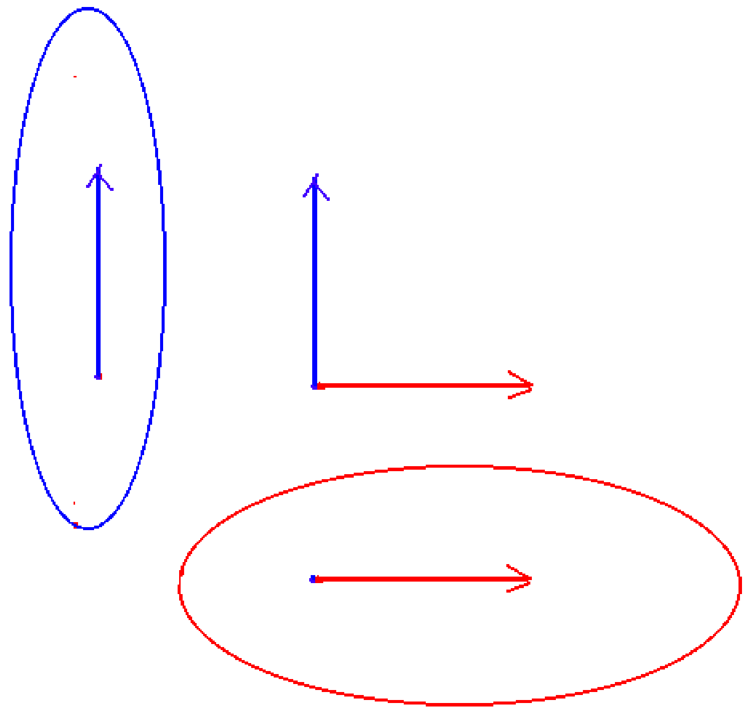
PRODUCTS



Def. The product of two Kripke frames

$(W,R_1,\ldots,R_n) \times (V,S_1,\ldots,S_m):=$

$(W\times V,R_{11},\ldots,R_{n1},S_{12},\ldots,S_{m2})$,

where

$(x_1,y_1)R_{i1}(x_2,y_2) \Leftrightarrow x_1R_ix_2 \;\&\; y_1=y_2$

$(x_1,y_1)S_{j2}\,(x_2,y_2) \Leftrightarrow x_1=x_2 \;\&\; y_1S_jy_2$

Def. The product of two modal logics

$$\mathbf{L_1 \times L_2 := L(\{F_1 \times F_2 \mid F_1 \vDash L_1,\ F_2 \vDash L_2\})}$$

AXIOMATIZATION PROBLEM: *to find axioms of* $\mathbf{L_1 \times L_2}$ *given the axioms of* $\mathbf{L_1,\ L_2}$

Def. The fusion of two modal logics with disjoint modalities

$$\mathbf{L_1 * L_2} := \text{the smallest logic containing } \mathbf{L_1} \cup \mathbf{L_2}$$

Def. The commutative join of two modal logics with disjoint modalities

$\boxminus_i$ $\mathbf{(1 \leqslant i \leqslant n)}$, $\boxbar_j$ $\mathbf{(1 \leqslant j \leqslant m)}$

$$\mathbf{[L_1,L_2] := L_1 * L_2 + \diamondminus_i \boxbar_k p \rightarrow \boxbar_k \diamondminus_i p + \boxminus_i \boxbar_k p \leftrightarrow \boxbar_k \boxminus_i p}$$

for any i, k

Remark. If the modalities are not disjoint, we can change them.

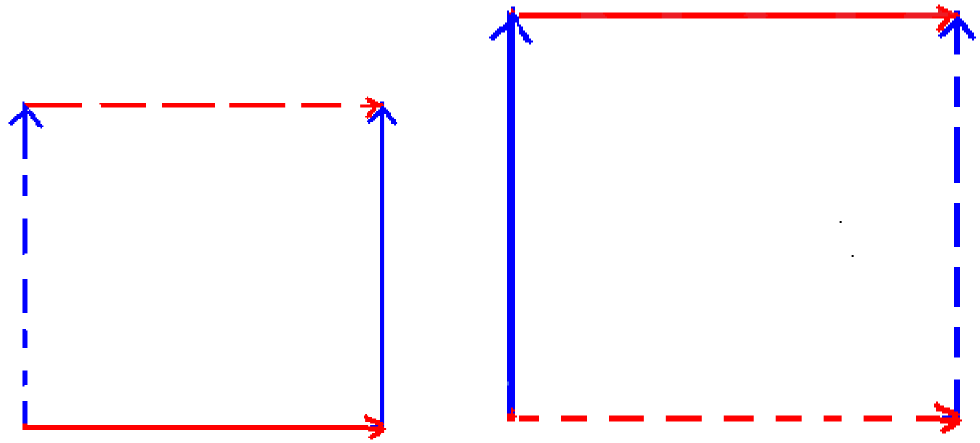
These are Salqvist formulas expressing the following properties of the relations in the product frame

$\Diamond_i \Box_k p \rightarrow \Box_k \Diamond_i p$:

$(R_{i1})^{-1} \circ S_{k2} \subseteq S_{k2} \circ (R_{i1})^{-1}$ (Church - Rosser property)

$\boxminus_i \boxbar_k p \leftrightarrow \boxbar_k \boxminus_i p$:

$R_{i1} \circ S_{k2} = S_{k2} \circ R_{i1}$ (commutativity)



Def. L_1, L_2 are product-matching if $L_1 \times L_2 = [L_1, L_2]$

SQUARES

For a class of frames C put

$$C^2 := \{F \times F \mid F \in C\}.$$

For a modal logic Λ put

$$\Lambda^2 := \Lambda \times \Lambda$$

Proposition 1 [Gabbay,Sh 2000]

$$\Lambda^2 = L(\{F \times F \mid F \models \Lambda\}).$$

("Squares of logics are determined by squares of frames".)

Proposition 2 [Gabbay,Sh 2000]

$$L_1 \times L_2 \text{ is embeddable in } (L_1 * L_2)^2.$$

("Products are reducible to squares".)

SEGERBERG SQUARES

These are square frames with additional functions. Krister Segerberg (1973) studied a special type - squares of frames with the universal relation.

He considered the following functions on squares.

$\sigma_{\bigcirc}$: $(x,y) \mapsto (y,x)$ (diagonal symmetry)

$\sigma_{\ominus}$: $(x,y) \mapsto (y,y)$ (the first diagonal projection)

σ_{ϕ}: $(x,y) \mapsto (x,x)$ (the second diagonal projection)

These functions can be associated with extra modal operators $\bigcirc$, $\ominus$, ϕ. So in square frames they are interpreted as follows:

$(x,y) \vDash \bigcirc A$ iff $(y,x) \vDash A$

$(x,y) \vDash \ominus A$ iff $(x,x) \vDash A$

$(x,y) \vDash \phi A$ iff $(y,y) \vDash A$

Remark. Segerberg used the notation $\otimes$ instead of $\bigcirc$.

Formally we define the Segerberg square of a frame $F=(W,R_1,\dots,R_n)$ as the (2n+3)-frame $F^{2\circledast} := (F^2, \sigma_{\bigcirc}, \sigma_{\ominus}, \sigma_{\varobar})$ (where $\sigma_{\bigcirc}, \sigma_{\ominus}, \sigma_{\varobar}$ are the functions on W^2 described above).

Respectively, the Segerberg square of an n-modal logic Λ is defined the logic of the Segerberg squares of its frames

$$\Lambda^{2\circledast} := L(\{F^{2\circledast} \mid F \vDash \Lambda\}).$$

TOMORROW (OR SUCCESSOR) LOGIC

$$\mathbf{SL} := \mathbf{K} + \Diamond p \leftrightarrow \Box p$$

(an equivalent form: $\mathbf{K} + \neg\Box p \leftrightarrow \Box\neg p$)

This well-known logic is also due to Segerberg (1967). It is complete w.r.t. the frame

(the successor relation on natural numbers).

Every logic of a frame with a functional accessibility relation is an extension of **SL**.

AXIOMATIZING SEGERBERG SQUARES

Soundness Every Segerberg square validates the following formulas

The corresponding semantic conditions for an arbitrary (2n+3)-frame

$(V, X_1, \dots, X_n, Y_1, \dots, Y_n, \mathbf{f}_{\bigcirc}, \mathbf{f}_{\ominus}, \mathbf{f}_{\phi})$

are in the right column; here fg denotes the composition of functions: (fg)(x)=f(g(x))

(I) The **SL**-axioms for the circles ○, ⊖, ⦶.

(II)

(Sg1) $\bigcirc\bigcirc p \leftrightarrow p$ $\quad$ $f_{\bigcirc} f_{\bigcirc} = 1$ (the identity function on V)

The "symmetry" $f_{\bigcirc}$ is an involution.

(Sg2) $\ominus\ominus p \leftrightarrow \ominus p$ $\quad$ $f_{\ominus} f_{\ominus} = f_{\ominus}$

(Sg2') ⦶⦶p $\leftrightarrow$ ⦶p $\quad$ $f_{⦶}f_{⦶} = f_{⦶}$

Both projections $\mathbf{f}_{\ominus}$, $\mathbf{f}_{⦶}$ are idempotent transformations of the square. In fact (Sg2') follows from (Sg1), (Sg2), (Sg3).

(Sg3) ○⊖p $\leftrightarrow$ ⦶p $\quad$ $f_{\ominus}f_{\circ} = f_{⦶}$

(Sg4) ⊖○p $\leftrightarrow$ ⊖p $\quad$ $f_{\circ}f_{\ominus} = f_{\ominus}$

In Segerberg squares (Sg4) means that the image of $\mathbf{f}_{\ominus}$ consists of self-symmetric points (or: every diagonal point is self-symmetric). But in the general case not all self-symmetric points are in $\mathbf{f}_{\ominus}$[V].

(Sg3), (Sg4) imply that
$\mathbf{f}_{\circ}\mathbf{f}_{\ominus}\mathbf{f}_{\circ} = \mathbf{f}_{⦶}$, i.e., the involution $\mathbf{f}_{\circ}$ conjugates the projections $\mathbf{f}_{\ominus}$ and $\mathbf{f}_{⦶}$.

(Sg3) shows that ⦶ is expressible in terms of ○, ⊖. **It also implies that**
$\mathbf{f}_{\ominus}$[V]= $\mathbf{f}_{⦶}$[V].

(Sg4') ⦶○p $\leftrightarrow$ ⦶p $\quad$ $f_{⦶}f_{\circ} = f_{⦶}$

This conjugate of (Sg4) is derivable from (Sg1), (Sg3), (Sg4).

(Sg5) ○⊟$_i$○p $\leftrightarrow$ ◫$_i$p $\quad$ $aR_{i1}b \Leftrightarrow f_{\circ}(a)R_{i2}f_{\circ}(b)$

(Sg5) $\bigcirc\boxminus_i\bigcirc p \leftrightarrow \boxbar_i p$ $aR_{i1}b \Leftrightarrow f_{\bigcirc}(a)R_{i2}f_{\bigcirc}(b)$

Symmetry is an isomorphism between R_{i1} and R_{i2}

(Sg6) $\ominus\boxminus_i(\boxbar_i p \rightarrow \obar p)$ $f_{\ominus}(a)R_{i1}b \Rightarrow bR_{i2}f_{\obar}(b)$

If $(y,y)R_{i1}(x,y)$ (i.e. yR_ix), then $(x,y)R_{i2}(x,x)$.

(Sg7) $\ominus p \rightarrow \boxminus_i\ominus p$ $aR_{i1}b \Rightarrow f_{\ominus}(a) = f_{\ominus}(b)$

Horizontally accessible points are in the same horizontal row.

(Sg8) $\boxbar_i \perp \leftrightarrow \ominus\boxbar_i \perp$ $(\exists b\ aR_{i2}b) \Leftrightarrow (\exists b\ f_{\ominus}(a)R_{i2}b)$

Vertical seriality is equivalent for (y,y) and (x,y).

The conjugates of (Sg6)-(Sg8) are derivable, so they are not written here.

Def. For a modal logic Λ, put

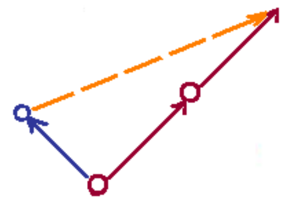
$[\Lambda, \Lambda]^{\circledast} :=$

$[\Lambda, \Lambda]$ + **SL**∗**SL**∗**SL** (for $\bigcirc$, $\ominus$, $\obar$) + {(Sg1),..., (Sg8)}.

Def. A universal Horn sentence is a first order sentence of the form $\forall x\ldots(\varphi(x,y,z)\rightarrow R(x,y))$, where φ **is** positive, R(x,y) is atomic.

Modal formulas corresponding to such sentences are conjunctions of formulas of the form

$$(\Diamond \ldots \Diamond)\Box p \to (\Box \ldots \Box)p$$



Def. A modal logic is Horn axiomatizable if it is axiomatizable by formulas that are either variable-free or correspond to universal Horn sentences.

Completeness theorem for products [Gabbay,Sh 1998]

If L_1, L_2 are Horn axiomatizable, then they are product-matching.

Theorem 1 (Completeness) If a logic Λ is Horn axiomatizable, then $\Lambda^{2\circledast} = [\Lambda, \Lambda]^{\circledast}$

Remark Segerberg himself axiomatized the logic **B** of all frames of the form $(W, W\times W)^{2\circledast}$. In this case (Sg8) becomes trivial and (Sg6) should be replaced with a stronger axiom: $\boxbar p \to \Phi p$. So Segerberg's logic is not a Segerberg square in our sense; it is a proper extension of $\mathbf{S5}^{2\circledast}$.

Sketch of the proof of Theorem 1

Step 1. $(\mathbf{K}_n)^{2\circledast} = [\mathbf{K}_n, \mathbf{K}_n]^{\circledast}$

Consider the case n=1. The logic L:= $[\mathbf{K}, \mathbf{K}]^{\circledast}$ is Sahhqvist, so it has the countable frame property, so it its determined by countable rooted L-frames. Let

$F=(W, R_1, R_2, \mathbf{f}_{\circ}, \mathbf{f}_{\ominus}, \mathbf{f}_{\phi})$ be such a frame.

Now the goal is to construct a p-morphism from a Segerberg square onto F. We use a "rectification game" similar to the one described in [Sh 2005] and originally motivated by the games from [Many-dimensional modal logics, 2003] and [Relation algebras by games, 2002].

Let $T_\omega = (\omega^*,<)$ be the standard countable intransitive irreflexive tree, where

ω^* is the set of all finite sequences in ω;

$\alpha < \beta$ iff $\exists n \in \omega\ \beta = \alpha n$.

Let $T_\omega + T_\omega$ be the disjoint union of its two copies:

$\{x\alpha \mid \alpha \in \omega^*\} \cup \{y\alpha \mid \alpha \in \omega^*\}$ with the relation $<$.

Consider the product frame

$(T_\omega + T_\omega)^2 = ((\omega^*+\omega^*)^2, S_1, S_2)$.

A *network* over F is a partial function from $(T_\omega + T_\omega)^2$ to F

$h: N \rightarrow V$

such that

- dom(h)=N is symmetric:

  $\sigma_\circ[N]=N$,

  $\sigma_\ominus[N] \subseteq N$.

- N does not have gaps:

$(\alpha,\beta)\in N$ & $(\alpha,\gamma)\in N$ & $\beta<^{+}\gamma$ & $\beta< \beta' \Rightarrow (\alpha,\beta')\in N$

($<^{+}$ is the transitive closure of $<$)

- h is monotonic:

  $aS_ib \Rightarrow h(a)R_ih(b)$,

  $h(\sigma_{\circ}(a))= f_{\circ}(h(a))$,

  $h(\sigma_{\ominus}(a))= f_{\ominus}(h(a))$.

The game between **A** and **E** constructs a countable increasing sequence of networks $h_0\subseteq h_1\subseteq\ldots$ according to the following rules.

1. $N_0=\{(x,y),(y,x),(x,x),(y,y)\}$, where

   $h_0(x,y)=u_0$, the root of F; then $h_0(y,x)$, $h_0(x,x)$, $h_0(y,y)$ are uniquely determined.

Remark. If u_0 is self-symmetric, we don't need two copies, the game can start from $N_0=\{(\lambda,\lambda)\}$, where λ is empty.

2. The (n+1)th move of **A** is of two types

<u>Lift enquiry</u> (a,u,j,v), where $a\in N_n$, $u=h_n(a)$, uR_jv

The response of **E** must be a network h_{n+1} extending h_n ch that $\exists b \in N_{n+1}(aS_j b$ &

THE FINITE MODEL PROPERTY

Def. A QT-formula is a modal formula of the form

$\Box_i p \rightarrow \Box_i^k p$ (generalized transitivity)

or

$\Diamond_i \Box_i p \rightarrow p$ **(symmetry)**

A QTC-logic is axiomatizable by formulas that are either variable-free or QT-formulas.

Notation $\mathbf{K}_{\pm n}$ **is the minimal n-temporal logic (axiomatized by** $\Diamond_i^{-1}\Box_i p \to p$, $\Diamond_i \Box_i^{-1} p \to p$)

The fmp for products [Sh 2005]

If $\mathbf{L}_2$ is a QTC-logic, then $\mathbf{K}_{\pm n} \times \mathbf{L}_2 = [\mathbf{K}_{\pm n}, \mathbf{L}_2]$ has the fmp.

Theorem 2 $(\mathbf{K}_n)^{2\circledast}$ has the fmp.

Conjecture $(\mathbf{K}_{\pm n})^{2\circledast}$ has the fmp.

THE PRODUCT FMP FOR SEGERBERG SQUARES

Def A logic $\Lambda^{2\circledast}$ has the product fmp if it is determined by finite Segerberg squares:

$$\Lambda^{2\circledast} = L(\{F^{2\circledast} \mid F \text{ is finite, } F \vDash \Lambda\}).$$

The product fmp for products [Gabbay, Sh 2002]

Every logic $K_{\pm n} \times K_m$ has the product fmp.

Theorem 3 $(K_n)^{2\circledast}$ has the product fmp.

Conjecture $(K_{\pm n})^{2\circledast}$ has the product fmp.

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Ideas for the proofs of Theorems 1,2,3.
Relation algebras