On Processes and Structures

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Theories:

1. Admissible sets and generalized computability.
2. Approximation spaces and domains.
3. Computable structures.

Questions:

1. How to define a measure (degree) of complexity of a given abstract structure, maybe uncountable?
2. For a given admissible set $\mathbb{A}$, what is computability on $\mathbb{A}$?
3. Which computabilities are generated by structures?
4. Which structures are generated by computabilities?

We propose the following approaches to the questions stated above:

1) The measure of (relative) complexity of a structure is given by its degrees in semilattices of Σ-degrees and degrees of presentability.
2) Computability on $\mathbb{A}$ is a family of its **components**, with each component defined as a pair: a family of objects — subsets of A, and a class of Σ-processes acting on them, with the property that every finite fragment of an output can be obtained using some finite fragments of the arguments and resources.
3) The jump of a component of computability on $\mathbb{A}$ is a structure with the domain consisting of its objects and the diagram is obtained by the **termination** of its processes.

Approaches and publications:

1. *Barwise J.* Admissible Sets and Structures. Springer. Berlin–Heidelberg–New York. – 1975.
2. *Ershov Yu. L.* Definability and Computability. Consultants Bureau, New York–London–Moscow. – 1996.
3. *Ershov Yu. L., Puzarenko V. G., and Stukachev A. I.* HF-Computability. In S. B. Cooper and A. Sorbi (eds.): Computability in Context: Computation and Logic in the Real World. Imperial College Press/World Scientific. 2011. P. 173–248.
4. *Stukachev A.* Effective model theory: an approach via Σ-definability // Lecture Notes in Logic. 2013. V. 41. P. 164–197.
5. *Stukachev A.* On processes and structures // Lecture Notes in Computer Science. 2013. V. 7921. P. 393–402.

Publications

I. On processes and structures

[M. Thes.] *Stukachev A. I.* Σ-admissible families over linear orderings // Algebra and Logic. 2002. V. 41. N. 2. P. 228–252.

2. *Stukachev A.* On mass problems of presentability // Lecture Notes in Computer Science. 2006. V. 3959. P. 774–784.
3. *Stukachev A. I.* Degrees of presentability of structures. I // Algebra and Logic. 2007. V. 46. N. 6. P. 763–788.
4. *Stukachev A. I.* Degrees of presentability of structures. II // Algebra and Logic. 2008. V. 47. N. 1. P. 108–126.
5. *Stukachev A. I.* A jump inversion theorem for the semilattices of Sigma-degrees of structures (Russian)// Siberian Electronic Mathematical Reports. 2009. V. 6. P. 182–190.
6. *Stukachev A.* On processes and structures // Lecture Notes in Computer Science. 2013. V. 7921. P. 393–402.

II. **On properties of generalized computability** (reducibilities $\leqslant_{\Sigma}$ and $\leqslant_{s\Sigma}$)

[B. Thes.] *Stukachev A. I.* Uniformization property in heredidary finite superstructures // Siberian Advances in Mathematics. 1997. V. 7. N 1. P. 123–132.

2. *Stukachev A. I.* On inner constructivizability of admissible sets (Russian) // Vestnik NGU. 2005. V. 5. N. 1. P. 69–76.
3. *Stukachev A.* Effective model theory: an approach via Σ-definability // Lecture Notes in Logic. 2013. V. 41. P. 164–197.
4. *Stukachev A. I.* On properties of $s\Sigma$-reducibility // Algebra and logic (to appear).

III. **On *c*-simple theories**

[Ph.D.] *Stukachev A. I.* Σ-definability in hereditarily finite superstructures and pairs of models // Algebra and Logic. 2004. V. 43. N. 4. P. 459–481.

2. *Stukachev A. I.* On Σ-degrees of uncountable models of *c*-simple theories // Siberian Mathematical Journal. 2010. V. 51. N. 3. P. 649–661.

3. *Stukachev A. I.* On *c*-simple theories // Algebra and Logic (to appear).

Plenary and invited talks:

1-3. Maltsev Meeting (Novosibirsk, Russia, 2004, 2005, 2010),
4. Computability in Europe 2007 (Siena, Italy, 2007) (joint with Yu. L. Ershov and V. G. Puzarenko),
5. Effective Mathematics of the Uncountable (New York, U.S.A., 2008),
6. Continuity, Computability, Constructivity: from Logic to Algorithms (Cologne, Germany, 2009),
7. Workshop on Computability Theory 2010 (Paris, France, 2010),
8. Algebra and Mathematical Logic (Kazan', Russia, 2011),
9. Definability in Computable Models (Chicago, U.S.A., 2012).

Contents

Definition (Yu. L. Ershov)

1. *A first-order theory T is called* **regular** *if it is decidable and model complete.*
2. *A first-order theory T is called c*-**simple** *(constructively simple) if it is decidable, model complete, ω-categorical, and has a decidable set of the complete formulas.*

Definition (S.)

A first-order theory T is called sc-**simple** *if it is decidable, submodel complete, ω-categorical, and has a decidable set of the complete formulas.*

Definition (A. J. Wilkie)

A first-order theory T is called **effectively model complete** *if, for any formula $\Phi(\overline{x})$ of the signature of T, there exists an $\exists$-formula $\Psi(\overline{x})$ which is equivalent (w.r.t. the theory T) to $\Phi(\overline{x})$, and $\Psi(\overline{x})$ can be found effectively from $\Phi(\overline{x})$.*

Let $\sigma' = \sigma \cup \{U^1, \in^2, \varnothing\}$ where σ is a finite signature.

Definition

*The class of Δ_0-**formulas** of signature σ' is the least one of formulas containing all atomic formulas of signature σ' and closed under $\wedge, \vee, \neg$, $\exists x \in y$ and $\forall x \in y$.*

Definition

*The class of Σ-**formulas** of signature σ' is the least one of formulas containing all Δ_0-formulas of signature σ' and closed under $\wedge, \vee$, $\exists x \in y$, $\forall x \in y$ and $\exists x$.*

Definition

The **axioms of KPU** *(of a signature σ') are the universal closures of the following formulas:*

Empty set: $\neg\exists x(x \in \varnothing) \wedge \neg U(\varnothing)$

Extensionality:
$(\neg U(a) \wedge \neg U(b)) \rightarrow (\forall x(x \in a \leftrightarrow x \in b) \rightarrow a = b)$

Foundation: $\exists x \varphi(x) \rightarrow \exists x[\varphi(x) \wedge \forall y \in x \neg\varphi(y)]$ *for all formulas* $\varphi(x)$ *in which* y *does not occur free*

Pair: $\exists a(x \in a \wedge y \in a)$

Union: $\exists b \forall y \in a \forall x \in y(x \in b)$

Δ_0**-Separation**: $\exists b \forall x(x \in b \leftrightarrow x \in a \wedge \varphi(x))$ *for all* Δ_0*-formulas* $\varphi(x)$ *in which* b *does not occur free*

Δ_0**-Collection**:
$\forall x \in a \exists y\, \varphi(x, y) \rightarrow \exists b \forall x \in a \exists y \in b\, \varphi(x, y)$ *for all* Δ_0*-formulas* $\varphi(x)$ *in which* b *does not occur free.*

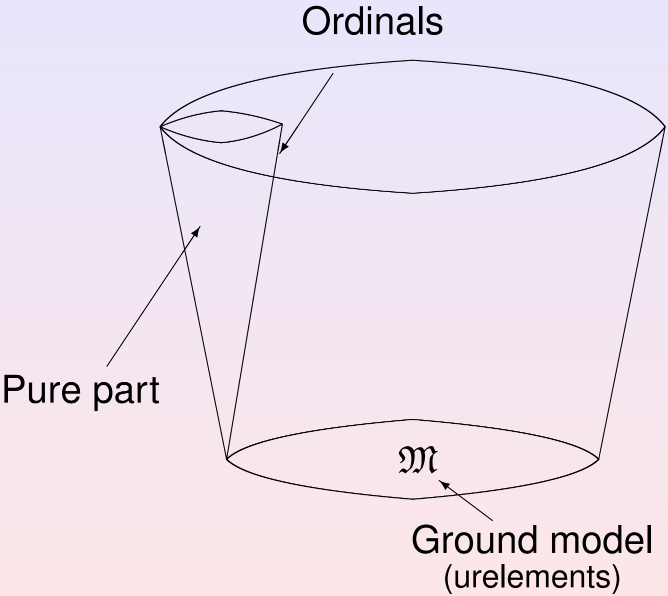
Let $\mathrm{Tran}(a)$ be the formula $\forall x \in a \forall y \in x (y \in a)$ and let

$$\mathrm{Ord}(a) \rightleftharpoons \mathrm{Tran}(a) \wedge \forall x \in a\, \mathrm{Tran}(x).$$

Definition

A structure $\mathbb{A}$ of signature σ' is called an **admissible set** *if*

1) $\mathbb{A} \models \mathrm{KPU}$
2) $\mathrm{Ord}\,\mathbb{A} = \{a \mid \mathbb{A} \models \mathrm{Ord}(a)\}$ *is wellfounded*


Ordinals
Pure part
$\mathfrak{M}$
Ground model
(urelements)

$\mathbb{HF}(\mathfrak{M})$

For a set M, consider the set $\mathrm{HF}(M)$ of hereditarily finite sets over M defined as follows: $\mathrm{HF}(M) = \bigcup_{n\in\omega} \mathrm{HF}_n(M)$, where

$\mathrm{HF}_0(M) = \{\varnothing\} \cup M$,
$\mathrm{HF}_{n+1}(M) = \mathrm{HF}_n(M) \cup \{a \mid a \text{ is a finite subset of } \mathrm{HF}_n(M)\}$.

For a structure $\mathfrak{M} = \langle M, \sigma^{\mathfrak{M}}\rangle$ of (finite or computable) signature σ, **hereditarily finite superstructure**

$$\mathbb{HF}(\mathfrak{M}) = \langle \mathrm{HF}(M); \sigma^{\mathfrak{M}}, U, \in, \varnothing\rangle$$

is a structure of signature σ' (with $\mathbb{HF}(\mathfrak{M}) \models U(a) \iff a \in M$).

Remark: in the case of infinite signature, we assume that σ' contains an additional relation $Sat(x, y)$ for atomic formulas under some fixed Gödel numbering.

Fact

$\mathbb{HF}(\mathfrak{M})$ *is the least admissible set over* $\mathfrak{M}$.

Σ-definability of structures in admissible sets

Let $\mathfrak{M}$ be a structure of a relational signature $\langle P_0^{n_0}, \ldots, P_k^{n_k} \rangle$ and let $\mathbb{A}$ be an admissible set.

Definition (Yu. L. Ershov 1985)

*$\mathfrak{M}$ is called **Σ-definable in $\mathbb{A}$** if there exist Σ-formulas $\varphi(x_0, y), \psi(x_0, x_1, y), \psi^*(x_0, x_1, y)$, $\varphi_0(x_0, \ldots, x_{n_0-1}, y), \varphi_0^*(x_0, \ldots, x_{n_0-1}, y), \ldots, \varphi_k(x_0, \ldots, x_{n_k-1}, y)$, $\varphi_k^*(x_0, \ldots, x_{n_k-1}, y)$ such that, for some parameter $a \in A$, $M_0 \leftrightharpoons \varphi^{\mathbb{A}}(x_0, a) \neq \varnothing$, $\eta \leftrightharpoons \psi^{\mathbb{A}}(x_0, x_1, a) \cap M_0^2$ is a congruence on $\mathfrak{M}_0 \leftrightharpoons \langle M_0, P_0^{\mathfrak{M}_0}, \ldots, P_k^{\mathfrak{M}_0} \rangle$, where*

$$P_k^{\mathfrak{M}_0} \leftrightharpoons \varphi_k^{\mathbb{A}}(x_0, \ldots, x_{n_k-1}) \cap M_0^{n_k}, \ k \in \omega,$$
$$\psi^{*\mathbb{A}}(x_0, x_1, a) \cap M_0^2 = M_0^2 \setminus \psi^{\mathbb{A}}(x_0, x_1, a),$$
$$\varphi_i^{*\mathbb{A}}(x_0, \ldots, x_{n_i-1}, a) \cap M_0^{n_i} = M_0^{n_i} \setminus \varphi_i^{\mathbb{A}}(x_0, \ldots, x_{n_i-1}) \text{ for all } i \leqslant k,$$

and the structure $\mathfrak{M}$ is isomorphic to the quotient structure $\mathfrak{M}_0 / \eta$.

Σ-definability of structures in admissible sets

Σ-definability of a model in an admissible set $\mathbb{A}$ is an extension (on computability in $\mathbb{A}$) of the notion of constructivizability of a model (in classical computability theory CCT).

For a countable structure $\mathfrak{M}$, the following are equivalent:

- $\mathfrak{M}$ is constructivizable (computable);
- $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\varnothing)$.

For arbitrary structures $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $\mathfrak{M} \leqslant_{\Sigma} \mathfrak{N}$ the fact that $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$.

For arbitrary cardinal α, let $\mathcal{K}_\alpha$ be the class of all structures (of computable signatures) of cardinality $\leqslant \alpha$. We define on $\mathcal{K}_\alpha$ an equivalence relation $\equiv_\Sigma$ as follows: for $\mathfrak{M}, \mathfrak{N} \in \mathcal{K}_\alpha$,

$$\mathfrak{M} \equiv_\Sigma \mathfrak{N} \text{ if } \mathfrak{M} \leqslant_\Sigma \mathfrak{N} \text{ and } \mathfrak{N} \leqslant_\Sigma \mathfrak{M}.$$

Structure

$$\mathcal{S}_\Sigma(\alpha) = \langle \mathcal{K}_\alpha / \equiv_\Sigma, \leqslant_\Sigma \rangle$$

is an upper semilattice with the least element, and, for any $\mathfrak{M}, \mathfrak{N} \in \mathcal{K}_\alpha$,

$$[\mathfrak{M}]_\Sigma \vee [\mathfrak{N}]_\Sigma = [(\mathfrak{M}, \mathfrak{N})]_\Sigma,$$

where $(\mathfrak{M}, \mathfrak{N})$ denotes the model-theoretic pair of $\mathfrak{M}$ and $\mathfrak{N}$.

Theorem (Yu. L. Ershov 1985)

$$\mathbb{C} \leqslant_\Sigma \mathbb{L},$$

for any dense linear ordering of size continuum.

Theorem (S. 2002)

$$\mathbb{HYP}(\mathfrak{M}) \equiv_\Sigma \mathfrak{M},$$

for any recursively saturated regular structure $\mathfrak{M}$.

Let $\mathfrak{M}$ be a structure of a computable signature and let $\mathbb{A}$ be an admissible set.

Definition

A **presentation** *of $\mathfrak{M}$ in $\mathbb{A}$ is any structure $\mathcal{C}$ such that $\mathcal{C} \cong \mathfrak{M}$ and the domain of $\mathcal{C}$ is a subset of A.*

We can treat (the atomic diagram of) a presentation $\mathcal{C}$ as a subset of A, using some Gödel numbering of the atomic formulas of the signature of $\mathfrak{M}$.

Definition

The **problem of presentability** *of $\mathfrak{M}$ in $\mathbb{A}$ is the family $\mathrm{Pr}_{\mathbb{A}}(\mathfrak{M})$ consisting of the atomic diagrams of all possible presentations of $\mathfrak{M}$ in $\mathbb{A}$:*

$$\mathrm{Pr}_{\mathbb{A}}(\mathfrak{M}) = \{\, \mathcal{C} \mid \mathcal{C} \text{ is a presentation of } \mathfrak{M} \text{ in } \mathbb{A} \,\}$$

Denote by $\underline{\mathfrak{M}}$ the set $\mathrm{Pr}_{\mathbb{HF}(\varnothing)}(\mathfrak{M})$ of all presentations of $\mathfrak{M}$ in the least admissible set.

A **mass problem** (Yu. T. Medvedev, 1955) is any set of total functions from ω to ω. A mass problem can be considered as a set of "solutions" (in form of functions from ω to ω) of some "informal problem".

Examples of mass problems: suppose $A, B \subseteq \omega$

1) the **problem of solvability** of a set A is the mass problem $\mathcal{S}_A = \{\chi_A\}$, where χ_A is the characteristic function of A
2) the **problem of enumerability** of a set A is the mass problem
$$\mathcal{E}_A = \{f : \omega \to \omega \mid \mathrm{rng}(f) = A\}$$
3) the **problem of separability** of sets A, B is the mass problem
$$\mathcal{P}_{A,B} = \{f : \omega \to 2 \mid f^{-1}(0) = A,\ f^{-1}(1) = B\}$$

Suppose $\mathcal{X}, \mathcal{Y} \subseteq P(A)$. $\mathcal{X}$ is **Medvedev reducible** to $\mathcal{Y}$ ($\mathcal{X} \leqslant_s^{\mathbb{A}} \mathcal{Y}$) if there exist binary Σ-operators F_0 and F_1 such that, for all $Y \in \mathcal{Y}$, $\langle Y, A \setminus Y \rangle \in \delta_c(F_0) \cap \delta_c(F_1)$ and, for some $X \in \mathcal{X}$, $X = F_0(Y, A \setminus Y)$ and $A \setminus X = F_1(Y, A \setminus Y)$.

$\mathcal{X}$ is **Dyment reducible** to $\mathcal{Y}$ ($\mathcal{X} \leqslant_e^{\mathbb{A}} \mathcal{Y}$) if there exists a unary Σ-operator F such that, for all $Y \in \mathcal{Y}$, $Y \in \delta_c(F)$ and $F(\mathcal{Y}) \subseteq \mathcal{X}$.

$\mathcal{X}$ is **Muchnik reducible** to $\mathcal{Y}$ ($\mathcal{X} \leqslant_w^{\mathbb{A}} \mathcal{Y}$) if, for any $Y \in \mathcal{Y}$, there exist binary Σ-operators F_0 and F_1 such that $\langle Y, A \setminus Y \rangle \in \delta_c(F_0) \cap \delta_c(F_1)$ and, for some $X \in \mathcal{X}$, $X = F_0(Y, A \setminus Y)$ and $A \setminus X = F_1(Y, A \setminus Y)$.

Let $\mathbb{A}$ be an admissible set, $r \in \{e, s, w, ew\}$. For structures $\mathfrak{M}, \mathfrak{N}$, denote by

$$\mathfrak{M} \leqslant_r^{\mathbb{A}} \mathfrak{N}$$

the fact that $\mathrm{Pr}_{\mathbb{A}}(\mathfrak{M}) \leqslant_r^{\mathbb{A}} \mathrm{Pr}_{\mathbb{A}}(\mathfrak{M})$.

For any $r \in \{e, s, w, ew\}$, $\leqslant_r$ denotes $\leqslant_r^{\mathbb{HF}(\varnothing)}$.

Let

$$\mathcal{K}_\Sigma(\mathfrak{M}) = \{\mathfrak{N} \mid \mathfrak{N} \leqslant_\Sigma \mathfrak{M}\}$$
$$\mathcal{K}_e^{\mathbb{A}}(\mathfrak{M}) = \{\mathfrak{N} \mid \mathfrak{N} \leqslant_e^{\mathbb{A}} (\mathfrak{M}, \bar{m}) \text{ for some } \bar{m} \in M^{<\omega}\}$$
$$\mathcal{K}_s^{\mathbb{A}}(\mathfrak{M}) = \{\mathfrak{N} \mid \mathfrak{N} \leqslant_s^{\mathbb{A}} (\mathfrak{M}, \bar{m}) \text{ for some } \bar{m} \in M^{<\omega}\}$$
$$\mathcal{K}_{ew}^{\mathbb{A}}(\mathfrak{M}) = \{\mathfrak{N} \mid \mathfrak{N} \leqslant_{ew}^{\mathbb{A}} \mathfrak{M}\}$$
$$\mathcal{K}_w^{\mathbb{A}}(\mathfrak{M}) = \{\mathfrak{N} \mid \mathfrak{N} \leqslant_w^{\mathbb{A}} \mathfrak{M}\}.$$

For any structure $\mathfrak{M}$ and any admissible set $\mathbb{A}$,

$$\mathcal{K}_\Sigma^{\mathbb{A}}(\mathfrak{M}) \subseteq \mathcal{K}_e^{\mathbb{A}}(\mathfrak{M}) \subseteq \mathcal{K}_s^{\mathbb{A}}(\mathfrak{M}) \subseteq \mathcal{K}_w^{\mathbb{A}}(\mathfrak{M}),$$

as well as $\mathcal{K}_e^{\mathbb{A}}(\mathfrak{M}) \subseteq \mathcal{K}_{ew}^{\mathbb{A}}(\mathfrak{M}) \subseteq \mathcal{K}_w^{\mathbb{A}}(\mathfrak{M})$ (S. 2007). In general, all these inclusions are proper in the case $\mathbb{A} = \mathbb{HF}(\varnothing)$ (I. Kalimullin 2009).

For any $r \in \{e, s, w, ew\}$, define the relation $\leqslant_r^{\mathbb{A}}$ on $\mathcal{K}$ in the following way: $\mathfrak{M} \leqslant_r^{\mathbb{A}} \mathfrak{N}$ if and only if $\mathcal{K}_r^{\mathbb{A}}(\mathfrak{M}) \subseteq \mathcal{K}_r^{\mathbb{A}}(\mathfrak{N})$, and let $\mathcal{S}_r^{\mathbb{A}} = \langle \mathcal{K}^{\mathbb{A}} / \equiv_r^{\mathbb{A}}, \leqslant_r^{\mathbb{A}} \rangle$ be a structure of **degrees of $\mathbb{A}$-presentability** corresponding to this reducibility relation.

Proposition (S. 2007)

Each of $\mathcal{S}_r^{\mathbb{A}}$, $r \in \{e, s, w, ew\}$, is an upper semilattice with 0, and there are natural embeddings ($\hookrightarrow$) and homomorphisms ($\rightarrow$)

$$\mathcal{D} \hookrightarrow \mathcal{D}_e \hookrightarrow \mathcal{S}_\Sigma^{\mathbb{A}} \rightarrow \mathcal{S}_e^{\mathbb{A}} \rightarrow \mathcal{S}^{\mathbb{A}}{}_s \rightarrow \mathcal{S}_w^{\mathbb{A}},$$

as well as $\mathcal{S}_e^{\mathbb{A}} \rightarrow \mathcal{S}_{ew}^{\mathbb{A}} \rightarrow \mathcal{S}_w^{\mathbb{A}}$

Theorem (Friedberg 1957)

Let $A \subseteq \omega$ be a set such that $\mathbf{0}' \leqslant_T A$. There exists a set $B \subseteq \omega$ such that

$$B' \equiv_T A.$$

Theorem (S. 2009)

Let $\mathfrak{A}$ be a structure such that $\mathbf{0}' \leqslant_\Sigma \mathfrak{A}$. There exists a structure $\mathfrak{B}$ such that

$$\mathfrak{B}' \equiv_\Sigma \mathfrak{A}.$$

Corollary

Let $\mathfrak{A}$ be a countable structure such that $0' \leqslant_\Sigma \mathfrak{A}$. Then there is a structure $\mathfrak{B}$ such that, for any $r \in \{e, s, w, ew\}$,

$$\mathfrak{B}' \equiv_r \mathfrak{A},$$

where the symbols e, s, w, ew denote Dyment, Medvedev, Muchnik, and non-uniform Dyment reducibilities, correspondingly.

Theorem (A.Soskova, I.Soskov 2009)

Let $\mathfrak{A}$ be a countable structure such that $\mathbf{0}' \leqslant_w \mathfrak{A}$. There exists a structure $\mathfrak{B}$ such that

$$\mathfrak{B}' \equiv_w \mathfrak{A}.$$

- Baleva, V. (2006) The jump operation for structure degrees, *Arch. Math. Logic* **45**, pp. 249–265.
- Soskov, I. N. and Soskova, A. A. (2009). A jump inversion theorem for the degree spectra, *J. Log. Comput.* **19**, pp. 199–215.
- Montalban, A. (2009). Notes on the jump of a structure, in CiE2009, *Lect. Notes Comput. Sci.* **5635**, pp. 372–378.

Definition (S.)

*Structure $\mathfrak{A}$ is called $s\Sigma$-**definable** in $\mathbb{HF}(\mathfrak{B})$ (denoted as $\mathfrak{A} \leqslant_{s\Sigma} \mathfrak{B}$) if $\mathfrak{A}$ is $\mathbb{HF}(\mathfrak{B})$-constructively generated, i.e., $A \subseteq \mathrm{HF}(B)$ is a Σ-subset of $\mathbb{HF}(\mathfrak{B})$, and all the signature relations and functions of $\mathfrak{A}$ are Δ-definable in $\mathbb{HF}(\mathfrak{B})$.*

Proposition (S. 2009)

For any structure $\mathfrak{A}$,

$$\mathfrak{A} <_{s\Sigma} \mathfrak{A}'.$$

Definition (S.2007)

1. A countable structure $\mathfrak{M}$ **has an *e*-degree** if there exists a presentation $\mathfrak{N} \cong \mathfrak{M}$ of a structure $\mathfrak{M}$ in $\mathbb{HF}(\varnothing)$, such that $\mathfrak{N} \leqslant_{s\Sigma} \mathfrak{M}$.
2. A countable structure $\mathfrak{M}$ **has a degree** if there exists a presentation $\mathfrak{N} \cong \mathfrak{M}$ of a structure $\mathfrak{M}$ in $\mathbb{HF}(\varnothing)$, such that $\mathfrak{N}^+ \leqslant_{s\Sigma} \mathfrak{M}$, where $\mathfrak{N}^+$ is the enrichment of $\mathfrak{N}$ by an unary predicate interpreted as $\mathfrak{N}$.

Definition (S. 2005)

HF-superstructure over a structure $\mathfrak{M}$ has **the rank of inner constructivizability** k, $k \in \omega$, if there exists a presentation $\mathfrak{N} \cong \mathbb{HF}(\mathfrak{M})$ of the HF-superstructure $\mathbb{HF}(\mathfrak{M})$ in $\mathbb{HF}(\mathfrak{M})$, such that $\mathfrak{N} \leqslant_{s\Sigma} \mathfrak{M}$ by means of the lements with the rank $\leqslant k$.

Proposition (S. 2005)

1. For any structure $\mathfrak{M}$, the rank of inner constructivizability of $\mathbb{HF}(\mathfrak{M})$ is $\leqslant 2$.
2. The rank of inner constructivizability of $\mathbb{HF}(\mathbb{R})$ is equal to 1.

Definition (S.)

A structure $\mathfrak{M}$ is called **quasiregular** *if*

$$\mathfrak{M}^{\text{Morley}} \equiv_{s\Sigma} \mathfrak{M},$$

where $\mathfrak{M}$ is the Morley expansion of $\mathfrak{M}$.

Let $\mathfrak{M}$ be a structure of signature σ, signature σ_* consists of all symbols from σ and function symbols $f_\varphi(x_1, \ldots, x_n)$ for all $\exists$-formulas $\varphi(x_0, x_1, \ldots, x_n) \in F_\sigma$. A structure $\mathfrak{M}_*$ of signature σ_* is called **existential Skolem expansion of** $\mathfrak{M}$ if $|\mathfrak{M}_*| = |\mathfrak{M}|$, $\mathfrak{M} \restriction_\sigma = \mathfrak{M}_* \restriction_\sigma$, and for any $\exists$-formula $\varphi(x_0, x_1, \ldots, x_n) \in F_\sigma$

$$\mathfrak{M}_* \models \forall x_1 \ldots \forall x_n (\exists x \varphi(x, x_1, \ldots, x_n) \rightarrow$$

$$\rightarrow \varphi(f_\varphi(x_1, \ldots, x_n), x_1, \ldots, x_n)).$$

Theorem (S. 1996, with corr. 2013)

If $\mathrm{Th}(\mathfrak{M})$ *is regular then* $\mathbb{HF}(\mathfrak{M})$ *has the uniformization property if and only if, for some well-defined existential Skolem expansion* $\mathfrak{M}_S$ *of* $\mathfrak{M}$,

$$\mathfrak{M}_S \equiv_{s\Sigma} \mathfrak{M}.$$

Theorem (S.)

If $\mathfrak{M}$ *is quasiregular then* $\mathbb{HF}(\mathfrak{M})$ *has the uniformization property if and only if, for some well-defined existential Skolem expansion* $\mathfrak{M}_S$ *of* $\mathfrak{M}$,

$$\mathfrak{M}_S \equiv_{s\Sigma} \mathfrak{M}.$$

Proposition (S.)

1. *If* $\mathfrak{M}$ *is quasiregular then* $\mathbb{HF}(\mathfrak{M})$ *has a universal* Σ*-function and the reduction property.*
2. *If* $\mathfrak{M}$ *is quasiregular and* $\mathbb{HF}(\mathfrak{M})$ *has the uniformization property, then* $\mathbb{HF}(\mathfrak{M})$ *is* Σ*-equivalent to the Moschovakis expansion* $\mathfrak{M}^*$*.*

Proposition (S. 1996)

For $\mathbb{R}$ and $\mathbb{Q}_p$, there exist well-defined sΣ-definable Skolem expansions.

Proof: use Σ-definable topology and topological properties of definable subsets

Corollary (S. 1996, indep. Korovina 1996 for $\mathbb{HF}(\mathbb{R})$)

$\mathbb{HF}(\mathbb{R})$ and $\mathbb{HF}(\mathbb{Q}_p)$ have the uniformization property and a universal Σ-function.

Proposition

$\mathbb{R}_{exp}$ is quasiregular and there exists a well-defined existential Skolem expansion $(\mathbb{R}_{exp})_S$ of $\mathbb{R}$ such that

$$(\mathbb{R}_{exp})_S \equiv_{s\Sigma} \mathbb{R}_{exp}.$$

.

Let $\mathcal{X} = (X, F, \leqslant)$ be an approximation space generated by an admissible set $\mathbb{A}$, i.e. $X \subseteq P(A)$, $F = A$, $\leqslant = \subseteq_{X \setminus U^{\mathbb{A}}} \cup =_{U^{\mathbb{A}}}$. We consider the following classes of Σ-processes on $\mathcal{X}$:

1) the class of Σ-*functions*, i.e., of $\mathbb{A}$-constructive processes of kind $f : F \to F$, with $\mathbb{A}$-finite arguments and values;
2) the class of Σ-*predicates*, i.e., of $\mathbb{A}$-constructive processes of kind $P : F \to X$, with $\mathbb{A}$-finite arguments;
3) the class of Σ-*operators*, i.e., of $\mathbb{A}$-constructive processes of kind $S : X \to X$, with arbitrary arguments and values.

Uniformization property is an example of relationships between Σ-predicates and Σ-functions.
Embeddability property is an example of relationships between Σ-predicates and Σ-operators.

Definition

Let $\mathbb{A}$ be an admissible set.

1) *A mapping $\alpha : A^n \to P(A)$ is called a* **Σ-predicate on $\mathbb{A}$** *if there is a Σ-formula $\varphi_\alpha(x_1, \dots, x_n, y)$ of signature $\sigma_\mathbb{A}$ (with no parameters from A) such that, for all $a_1, \dots, a_n, b \in A$, $b \in \alpha(a_1, \dots, a_n)$ if $\mathbb{A} \models \varphi_\alpha(a_1, \dots, a_n, b)$ (φ_α is called a Σ-specification, or Σ-presentation, of α).*
2) *A mapping $\beta : P(A)^n \to P(A)$ is called a* **Σ-operator on $\mathbb{A}$** *if there is a Σ-formula $\varphi_\beta(x_1, \dots, x_n, y)$ of signature $\sigma_\mathbb{A}$ (with no parameters from A) such that, for all $S_1, \dots, S_n \in P(A)$, and $b \in A$*

$$b \in \beta(S_1, \dots, S_n) \text{ iff } \exists a_1 \subseteq S_1, \dots, \exists a_n \subseteq S_n$$

$$\text{s.t. } \mathbb{A} \models \varphi_\beta(a_1, \dots, a_n, b)$$

(here it is assumed that $a_1, \dots, a_n \in A^*$). Again, φ_β is called a Σ-specification, or Σ-presentation, of β.

We assume that if $\mathbb{A}$ is fixed, $\leqslant$ denotes $\subseteq_{A^*} \cup =_{U^{\mathbb{A}}}$.

Definition

Let $\mathbb{A}$ be an admissible set, and let $m, n \in \omega$. A mapping γ from $A^m \times (A \cup P(A))^n$ to $P(A)$ is called a **Σ-process on $\mathbb{A}$** ($(m, n) - \textit{process}$) if there is a Σ-formula $\varphi_\gamma(x_1, \ldots, x_m, y_1, \ldots, y_n, z)$ of signature $\sigma_{\mathbb{A}}$ (with no parameters from A) such that, for all $a_1, \ldots, a_m \in A$, $x_1, \ldots, x_m \in A \cup P(A)$, $c \in A$,

$$c \in \gamma(\bar{a}, \bar{x}) \text{ iff } \exists b_1 \leqslant x_1, \ldots, \exists b_n \leqslant x_n \text{ s.t. } \mathbb{A} \models \varphi_\gamma(\bar{a}, \bar{b}, c).$$

Formula φ_γ is called a Σ-*specification*, or Σ-*presentation*, of γ. The set of all Σ-presentations of a given process γ is denoted by $Pres_\Sigma(\gamma)$.

We denote by $\mathcal{F}_\Sigma(\mathbb{A})$ the class of all Σ-predicates on $\mathbb{A}$, by $\mathcal{O}_\Sigma(\mathbb{A})$ the class of all Σ-operators on $\mathbb{A}$, and by $\mathcal{P}_\Sigma(\mathbb{A})$ the class of all Σ-processes on $\mathbb{A}$ (hence, $\mathcal{P}_\Sigma(\mathbb{A}) \supseteq \mathcal{F}_\Sigma(\mathbb{A}) \cup \mathcal{O}_\Sigma(\mathbb{A})$).

Definition

Let $\mathbb{A}$ be an admissible set and let $\mathcal{C} \subseteq \mathcal{P}_\Sigma(\mathbb{A})$ be a class of Σ-processes on $\mathbb{A}$. A family $\mathcal{S} \subseteq A \cup P(A)$ is called **Σ-admissible relative to** $\mathcal{C}$ if

1) $\mathcal{S}$ is closed relative to processes from $\mathcal{C}$: for any (m, n)-process $\alpha \in \mathcal{C}$,

$$\forall a_1, \ldots, a_m \in A \forall x_1, \ldots, x_n \in \mathcal{S}\ \alpha(\bar{a}, \bar{x}) \in \mathcal{S};$$

2) processes from $\mathcal{C}$ are *strongly continuous* on elements from $\mathcal{S}$: for any (m, n)-process $\alpha \in \mathcal{C}$,

$$\forall a_1, \ldots, a_m \in A \forall x_1, \ldots, x_n \in \mathcal{S} \forall c \in A(c \leqslant \alpha(\bar{a}, \bar{x}) \text{ iff}$$

$$\exists b_1 \in A \ldots \exists b_n \in A(b_1 \leqslant x_1 \wedge \ldots \wedge b_n \leqslant x_n \wedge c \leqslant \alpha(\bar{a}, \bar{b}))).$$

Definition

Let $\mathbb{A}$ be an admissible set. By a **computability component on** $\mathbb{A}$ we mean a pair $(\mathcal{S}, \mathcal{C})$, where

1) $A \subseteq \mathcal{S} \subseteq P(A)$ is a Σ-admissible family relative to $\mathcal{C}$, and
2) $\mathcal{F}_\Sigma(\mathbb{A}) \subseteq \mathcal{C} \subseteq \mathcal{P}_\Sigma(\mathbb{A})$ is a class of Σ-processes on $\mathbb{A}$ which is closed under superposition.

For an admissible set $\mathbb{A}$, by **computability on** $\mathbb{A}$ we mean the family $\mathrm{Com}(\mathbb{A})$ of all computability components on $\mathbb{A}$:

$$\mathrm{Com}(\mathbb{A}) = \{(\mathcal{S}, \mathcal{C}) \mid (\mathcal{S}, \mathcal{C}) \text{ is a computability component on } \mathbb{A}\}.$$

To demonstrate the usefulness of these new notions, we prove a strengthening of the result of A.S.Morozov (2004) which states that a certain reducibility on admissible sets implies an embedding of computable objects (i.e., Σ-predicates) on them. The reducibility on admissible sets was defined by A.S.Morozov (2004)as a modification of the notion of **Σ-definability** of a structure in an admissible set introduced by Yu.L.Ershov (1985).

- Morozov, A. S. (2004). On the relation of Σ-reducibility between admissible sets, *Sib. Math. J.* **45**, 3, pp. 522–535.
- Khisamiev, A. N. (2004). On the Ershov Upper Semilattice $\mathfrak{L}_E$, *Sib. Math. J.* **45**, 1, pp. 173–187.
- Puzarenko, V. G. (2009). About a certain reducibility on admissible sets, *Sib. Math. J.* **50**, 2, pp. 330–340.

Definition (A.S. Morozov 2004)

Let $\mathbb{A}$ and $\mathbb{B}$ be admissible sets. $\mathbb{A}$ **is Σ-reducible to** $\mathbb{B}$ (denoted $\mathbb{A} \sqsubseteq_\Sigma \mathbb{B}$) if there is an onto mapping $\nu : B \twoheadrightarrow A$ such that

1) ν is a $\mathbb{B}$-constructivization of $\mathbb{A}$ as a structure;
2) there is a binary Σ-predicate E on $\mathbb{B}$ s.t. $pr_1(E) = B$ and, for all $b, c \in B$,

$$\langle b, c \rangle \in E \text{ implies } \nu(b) = \{\nu(z) | z \in c\}.$$

Definition

If, for admissible sets $\mathbb{A}, \mathbb{B}$, there exist mappings $\nu : B \twoheadrightarrow A$ and $\mu : Pres_\Sigma(\mathcal{P}_\Sigma(\mathbb{A})) \to Pres_\Sigma(\mathcal{P}_\Sigma(\mathbb{B}))$ such that μ is computable and, for every $(\mathcal{S}, \mathcal{C}) \in \mathrm{Com}(\mathbb{A})$, there exists $(\mathcal{S}', \mathcal{C}') \in \mathrm{Com}(\mathbb{B})$ such that

$$(\nu^{-1}(\mathcal{S}), \mu(Pres(\mathcal{C}))) \text{ is isomorphic to } (\mathcal{S}', \mathcal{C}'),$$

we say that $\mathrm{Com}(\mathbb{A})$ **is Σ-embeddable into** $\mathrm{Com}(\mathbb{B})$.

Theorem

Let $\mathbb{A}, \mathbb{B}$ be admissible sets. If $\mathbb{A} \sqsubseteq_\Sigma \mathbb{B}$ then $\mathrm{Com}(\mathbb{A})$ is Σ-embeddable into $\mathrm{Com}(\mathbb{B})$.

Σ-Jump of a Structure as the Jump of the Minimal Component of HF-Computability

Definition

Let $\mathbb{A}$ be an admissible set, and let $(\mathcal{S}, \mathcal{C})$ be a computability component on $\mathbb{A}$. The **jump of** $(\mathcal{S}, \mathcal{C})$ is the structure $J_{\mathbb{A}}(\mathcal{S}, \mathcal{C})$ with domain $\mathcal{S}$ and atomic diagram consisting of a unary predicate distinguishing the set A of finite objects and the termination $t(\mathcal{C})$ of processes from $\mathcal{C}$.

This extends in a natural way all existing definitions of jump operations defined on subsets of natural numbers or on structures. Indeed, in the last case, we use the fact that every structure generates the least admissible set containing it — HF-superstructure. If we take the least computability component on that HF-superstructure and terminate all its processes (i.e., all Σ-predicates), we get the structure which is called *Σ-jump* of the original one.

Definition

1) *Termination* of a (partial) functional $\alpha : F^n \to X$ is a total function $\alpha_t : F^{n+1} \to \{0, 1\}$ defined as follows: for any $\bar{a} \in F^n$, $b \in F$,

$$\alpha_t(\bar{a}, b) = 1 \text{ iff } b \leqslant \alpha(\bar{a}).$$

2) *Termination* of a (partial) operator $\beta : X^n \to X$ is a total function $\beta_t : X^{n+1} \to \{0, 1\}$ defined as follows: for any $\bar{a} \in X^n$, $b \in X$,

$$\beta_t(\bar{a}, b) = 1 \text{ iff } b = \beta(\bar{a}).$$

3) *Termination* of a (partial) process $\gamma : F^m \times X^n \to X$, $n > 0$, is a total function $\beta_t : F^m \times X^{n+1} \to \{0, 1\}$ defined as follows: for any $\bar{a} \in F^m$, $\bar{b} \in X^n$, $c \in X$,

$$\gamma_t(\bar{a}, \bar{b}, c) = 1 \text{ iff } c = \gamma(\bar{a}, \bar{b}).$$

The formal definition of Σ-jump is as follows:

Definition

Let $\mathfrak{A}$ be a structure. By **Σ-jump**, or *minimal Σ-jump*, of $\mathfrak{A}$, we mean the structure

$$\mathfrak{A}' = (X; F, \mathcal{T}),$$

with the domain $X = HF(A)$, and relations $F = HF(A)$ (domain consists of finite objects only, so the unary relation F is trivial in this case and usually skipped), and $\mathcal{T} = t(\mathcal{F}_\Sigma(\mathbb{HF}(\mathfrak{A})))$ as the termination of all Σ-predicates on $\mathbb{HF}(\mathfrak{A})$ (denoted by $\Sigma\text{-Sat}_{\mathbb{HF}(\mathfrak{A})}$).

Definition

Let $\mathfrak{A}$ be a structure. By *$P\Sigma$-jump*, or *maximal Σ-jump*, of $\mathfrak{A}$, we mean the structure

$$\mathfrak{A}^\diamond = (X; F, \mathcal{T}),$$

with the domain $X = HF(A) \cup P(HF(A))$, and the atomic diagram consisting of relations $F = HF(A)$ distinguishing finite objects, and $\mathcal{T}$ as the termination of all Σ-processes on $\mathbb{HF}(\mathfrak{A})$.

It is easy to note that $P\Sigma$-jump is indeed a jump with respect to $\leqslant_\Sigma$, because immediately from cardinality reasons we get that, for any structure $\mathfrak{A}$,

$$\mathfrak{A} <_\Sigma \mathfrak{A}^\diamond.$$

A natural question is an analogue of Jump Inversion Theorem for $P\Sigma$-jump. We start from investigating the Σ-degree of $0^\diamond$.

Definition

Let $\mathbb{R}$ denote the set of real numbers. We consider the following structures:

1) algebraical field of reals $\mathcal{R} = (\mathbb{R}, +, \times, 0, 1, =)$;

2) topological field of reals

$$\mathcal{R}_o = (\mathbb{R}, \Gamma^A_+, \Gamma^B_+, \Gamma^A_\times, \Gamma^B_\times, 0, 1, <),$$

where $\Gamma^A_+ = \{\langle x, y, z\rangle \in \mathbb{R}^3 | x + y < z\}$,
$\Gamma^B_+ = \{\langle x, y, z\rangle \in \mathbb{R}^3 | z < x + y\}$ (similar definitions for $\Gamma^A_\times$, $\Gamma^B_\times$).

A positive version of Σ-definability: for a structure $\mathfrak{M}$ and an admissible set $\mathbb{A}$, $\mathfrak{M}$ is **Σ^+-definable in $\mathbb{A}$** if there exist a computable sequence of Σ-formulas $\Phi(x_0, y), \Phi_0(x_0, \ldots, x_{n_0-1}, y), \Phi_1(x_0, \ldots, x_{n_1-1}, y), \ldots$ such that, for some parameter $a \in A$ and an onto mapping $\nu : \Phi^{\mathbb{A}}(x_0, a) \twoheadrightarrow M$, for every $i \in \omega$ and every $a_0, \ldots, a_{n_i-1} \in \Phi^{\mathbb{A}}(x_0, a)$,

$$\mathbb{A} \models \Phi_i(a_0, \ldots, a_{n_i-1}, a) \iff \mathfrak{M} \models P_i(\nu(a_0), \ldots, \nu(a_{n_i-1})).$$

Again, for structures $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $\mathfrak{M} \leqslant^+_\Sigma \mathfrak{N}$ the fact that $\mathfrak{M}$ is Σ^+-definable in $\mathbb{HF}(\mathfrak{N})$. It should be noted, however, that $\leqslant^+_\Sigma$ is transitive only in case when all structures are treated positively in the sense that their atomic diagrams are not necessarily closed under negations.

Theorem

$\mathcal{R}_o \leqslant_{\Sigma}^{+} 0^{\diamond}$.

Theorem

$\mathcal{R} \leqslant_{\Sigma} (0^{\diamond})'$.

Open Questions

1. What is an analogue of Jump Inversion for a given computability component of HF-computability over 0 or any given structure?
2. What is an analogue of Jump Inversion for a given computability component of $\mathbb{A}$-computability? This question is especially interesting for the least computability component of $\mathbb{HYP}(\mathfrak{M})$-computability.
3. Is $0^\diamond \leqslant^+_\Sigma \mathcal{R}_0$? This would mean that in the maximal component of HF-computability over 0 holds an analogue of the Matijasevich Theorem. Also, is it natural to ask whether or not $(0^\diamond)' \leqslant_\Sigma \mathcal{R}$.

Conjecture (Yu.L. Ershov, 1998)

Suppose a theory T has an uncountable model which is Σ-definable in $\mathbb{HF}(\mathfrak{M})$, for some structure $\mathfrak{M}$ with a c-simple theory. Then T has an uncountable model which is Σ-definable in $\mathbb{HF}(\mathbb{L})$ for some $\mathbb{L} \models T_{DLO}$.

The formal consequence of this conjecture is

Conjecture

Any c-simple theory has an uncountable model which is Σ-definable in $\mathbb{HF}(\mathbb{L})$ for some $\mathbb{L} \models T_{DLO}$.

Theorem (S. 2002)

There exists a sc-simple theory (of infinite signature) such that none of its uncountable models is Σ-definable in $\mathbb{HF}(\mathbb{L})$ for any $\mathbb{L} \models T_{DLO}$.

Theorem (S. 2010)

Let T be a sc-simple theory of finite signature. Then there exists an uncountable model $\mathfrak{M}$ of T such that $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathbb{L})$, $\mathbb{L} \models T_{DLO}$.

Definition

For arbitrary structures $\mathfrak{M}$ *and* $\mathfrak{N}$ *a set* $I \subseteq M^k \cap N$ *is called a set of* $\mathfrak{M}$**-indiscernibles in** $\mathfrak{N}$ *(of dimension* k*) if for any tuples* $\bar{i}, \bar{i}' \in I^{<\omega}$ *of the same length*

$$\langle \mathfrak{M}, \bar{i} \rangle \equiv \langle \mathfrak{M}, \bar{i}' \rangle \text{ implies } \langle \mathfrak{N}, \bar{i} \rangle \equiv \langle \mathfrak{N}, \bar{i}' \rangle.$$

Proposition (S. 2002)

If $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$, $\mathfrak{M}$ is uncountable and $\mathfrak{N}$ is ω-saturated and locally constructivizable of level ω, then there are computable $\mathfrak{M}' \equiv \mathfrak{M}$ and computable $\mathfrak{N}' \equiv \mathfrak{N}$ s.t. there is an infinite computable set of $\mathfrak{M}^*$-indiscernibles in $\mathfrak{N}'$, where $\mathfrak{M}^*$ is an expansion of $\mathfrak{M}'$ by a finite number of constants.

Theorem (S. 2002)

Let T be a c-simple theory and $\mathfrak{M}$ be any computable model of T.

i) *If T has an uncountable model Σ-definable in $\mathbb{HF}(\mathbb{L})$ for some $\mathbb{L} \models T_{DLO}$ then there exists an infinite computable set of order indiscernibles in $\mathfrak{M}$ (of dimension 1).*

ii) *If T has an uncountable model Σ-definable in $\mathbb{HF}(\mathbb{S})$ for some infinite set $\mathbb{S}$ then there exists an infinite computable set of total indiscernibles in $\mathfrak{M}$ (of dimension 1).*

Definition

A theory T is called discrete *if there exists $n_0 \in \omega$ such that, for any $n \in \omega$, any n-type of T is exactly determined by its n_0-subtypes.*

Observation

- *Any sc-simple theory of a finite signature is discrete.*
- *If a c-simple theory T of a signature σ is discrete then there is a finite $\sigma_0 \subseteq \sigma$ and a finite expansion $\sigma' \supseteq \sigma_0$, definable in σ_0, such that T is sc-simple in σ'.*

Theorem

There exists a c-simple theory of a finite signature such that none of its uncountable models is Σ-definable in $\mathbb{HF}(\mathbb{L})$ *for any* $\mathbb{L} \models T_{DLO}$.

Thank you!

Let $\mathfrak{M}$ be a structure of relational computable signature $\langle P_0^{n_0}, \ldots, P_k^{n_k}, \ldots \rangle$ and let $\mathbb{A}$ be an admissible set.

Definition (Yu.L. Ershov)

$\mathfrak{M}$ is called **Σ-definable in $\mathbb{A}$** *if there exists a computable sequence of Σ-formulas $\varphi(x_0, y), \psi(x_0, x_1, y), \psi^*(x_0, x_1, y)$, $\varphi_0(x_0, \ldots, x_{n_0-1}, y), \varphi_0^*(x_0, \ldots, x_{n_0-1}, y), \ldots, \varphi_k(x_0, \ldots, x_{n_k-1}, y)$, $\varphi_k^*(x_0, \ldots, x_{n_k-1}, y), \ldots$ such that, for some parameter $a \in A$, $M_0 \rightleftharpoons \varphi^{\mathbb{A}}(x_0, a) \neq \varnothing$, $\eta \rightleftharpoons \psi^{\mathbb{A}}(x_0, x_1, a) \cap M_0^2$ is a congruence on the structure $\mathfrak{M}_0 \rightleftharpoons \langle M_0, P_0^{\mathfrak{M}_0}, \ldots, P_k^{\mathfrak{M}_0}, \ldots \rangle$, where*

$$P_k^{\mathfrak{M}_0} \rightleftharpoons \varphi_k^{\mathbb{A}}(x_0, \ldots, x_{n_k-1}) \cap M_0^{n_k}, \ k \in \omega,$$
$$\psi^{*\mathbb{A}}(x_0, x_1, a) \cap M_0^2 = M_0^2 \setminus \psi^{\mathbb{A}}(x_0, x_1, a),$$

$\varphi_k^{*\mathbb{A}}(x_0, \ldots, x_{n_k-1}, a) \cap M_0^{n_k} = M_0^{n_k} \setminus \varphi_k^{\mathbb{A}}(x_0, \ldots, x_{n_k-1})$ for all $k \in \omega$, and the structure $\mathfrak{M}$ is isomorphic to the quotient structure $\mathfrak{M}_0 / \eta$.

For a countable structure $\mathfrak{M}$, the following are equivalent:

- $\mathfrak{M}$ is constructivizable (computable);
- $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\varnothing)$.

For arbitrary structures $\mathfrak{M}$ and $\mathfrak{N}$, we denote by $\mathfrak{M} \leqslant_\Sigma \mathfrak{N}$ the fact that $\mathfrak{M}$ is Σ-definable in $\mathbb{HF}(\mathfrak{N})$.

Definition

P-domain is a triple $\mathcal{X} = \langle X, F, \leqslant\rangle$, where X is a set of *objects*, $F \subseteq X$ is a set of *finite objects*, $\leqslant$ is a partial order on X.

Usually, X is a *completion* of F w.r.t. topology with basis consisting of the upper cones generated by finite elements:

$$\forall x \in X(x = \sup\{a \in F | a \leqslant x\}).$$

Definition

For a P-domain $\mathcal{X}$, we consider *processes* of two kinds:

functionals ($F^m \to X$),

operators ($X^n \to X$).

Definition

Operator $\alpha : X \to X$ *is continuous in* $\mathcal{X}$ *if, for every* $x \in X$ *and every* $a \in F$,

$$a \leqslant \alpha(x) \text{ iff } a \leqslant \alpha(b) \text{ for some } b \leqslant x,\ b \in F.$$

Specification of a process $\alpha : X^n \to X$ is a mapping $\alpha_S : F^{n+2} \to \{0,1\}$ such that, for every $\bar{x} \in X^n$ and $y \in X$,

$\alpha(\bar{x}) = y$ iff $\alpha_S(\bar{a}, b, c) = 1$ for some $\bar{a} \in F^n, b, c \in F$ s.t. $\bar{a} \leqslant \bar{x}, b \leqslant y$.

For an admissible set $\mathbb{A}$, *constructive processes* are exactly those specified by Δ_0-formulas in $\mathbb{A}$, with $F = A$ and $X \subseteq P(A)$ is a Σ-admissible family over $\mathbb{A}$.
The last argument of the specification stands for the resource component necessary for performing the process ('space'+'time').