PRIMITIVE, UNIMODULAR, and MEASURE PRESERVING SYSTEMS OF ELEMENTS (on the varieties of solvable groups)

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In the report connections among will be defined:

• primitive systems of elements (primitive elements);

• measure preserving systems of elements (measure preserving elements);

• unimodular systems of elements (unimodular elements).

In the beginning of my report I will consider these connections for absolutely free group and then for groups that are free in some solvable varieties and variety of profinite metabelian groups.

THE FOX DERIVATIONS

Fix some bases $X = \{x_1, \ldots, x_r\}$ in a free group F_r. Let $\mathbb{Z}(F_r)$ be the integral group ring of the group F_r and Δ its augmentation ideal; that is, the kernel of natural homomorphism

$$\varepsilon : \mathbb{Z}(F_r) \longrightarrow \mathbb{Z}.$$

The ideal Δ is a free right (left) $\mathbb{Z}(F_r)$–module with a free basis $\{(x_i - 1) \,|\, 1 \leq i \leq r\}$:

$$\Delta = \sum_{i=1}^{r} (x_i - 1)\mathbb{Z}(F_r).$$

For $v \in \Delta$ the right Fox derivations $\partial_i(v)$ in the basis X are projections of the element v on the corresponding direct summands.
For each $u \in \mathbb{Z}(F_r)$ the element $u - \varepsilon(u)$ belongs to Δ. Thus, any element $u \in \mathbb{Z}(F_r)$ can be written in the form

$$u - \varepsilon(u) = \sum_{i=1}^{r} (x_i - 1)\partial_i(u)$$

uniquely.
The elements $\partial_i(u)$ are called the right Fox derivatives of $u \in \mathbb{Z}(F_r)$ in the basis X.

For each $u, v \in \mathbb{Z}(F_r)$ we have

$$\partial_i(u+v) = \partial_i(u) + \partial_i(v).$$

$$\partial_i(uv) = \partial_i(u)v + \varepsilon(u)\partial_i(v).$$

It follows that

$$\partial_i(x_j) = \delta_{i,j},$$

where $\delta_{i,j}$ is the Kronecker's symbol.

PRIMITIVE, MEASURE PRESERVING, AND UNIMODULAR ELEMENTS
(on the variety of all groups.)

Definition. An element v of a free group F_r is called unimodular if

$$\partial_1 v \cdot \alpha_1 + \ldots + \partial_r v \cdot \alpha_r = 1$$

for some $\alpha_1, \ldots, \alpha_r \in \mathbb{Z}(F_r)$.

Definition. An element v of a free group F_r is called primitive if it is a subset of some set of free generators in F_r.

The property of an element v in the free group F_r to be unimodular is independent of the basis in which v is represented. This means that

$$\{\ \textit{primitive element}\} \quad \Longrightarrow \quad \{\ \textit{unimodular} \quad \textit{element}\}.$$

Indeed, let $v = x_1$ be a primitive element. Then

$$\partial_1 v = 1,\ \partial_2 v = \ldots = \partial_r v = 0.$$

Hence the ideal generated by $\partial_1 v, \ldots, \partial_r v$ is equal to $\mathbb{Z}(F_r)$. In other words, the element v is unimodular.

Definition of a verbal map.

Let $v = v(x_1, \ldots, x_r)$ be an element of the free group F_r. Consider a finite group G. Denote by G^r the direct product $\underbrace{G \times \ldots \times G}_{r}$. Given v, define the verbal map

$$\varphi_v : G^r \longrightarrow G$$

assigning $g = v(g_1, \ldots, g_r)$ to $\overline{g} = (g_1, \ldots, g_r) \in G^r$, i.e. g is the value of the word v at $\overline{g}$.

A measure preserving element.

Define a uniform distribution corresponding to a random choice of elements $\overline{g}$ in G^r, i.e. every $\overline{g} \in G^r$ is chosen with probability $|G|^{-r}$.

Definition. An element v is called measure preserving on G if each $g \in G$ appears as image under φ_v exactly $|G|^{r-1}$ times, i.e., with probability $|G|^{-1}$.

Definition. An element v that preserves measure on every finite group G is called a measure preserving (on the variety of all groups).

Trivially,

$$\{\textit{a primitive element}\} \implies \{\textit{a measure preserving element}\}$$

Actually, note that the property of an element v in the free group F_r to preserve measure is independent of the basis in which v is represented. Therefore let $v = x_1$. Then

$$\varphi_v(g_1, \ldots, g_r) = v(g_1, \ldots, g_r) = g_1.$$

This means that each $g_1 \in G$ has exactly $|G|^{r-1}$ preimages under φ_v, i.e. appears with probability $|G|^{-1}$.

SOME KNOWN FACTS AND CONJECTURES
OF THE PRIMITIVE, MEASURE PRESERVING,
AND UNIMODULAR ELEMENTS
(on the variety of all groups)

Conjecture 1. An element $v \in F_r$ is primitive if and only if it preserves measure.

This Conjecture was formulated by D.Puder in his paper ”On primitive words II: measure preservation”.
He write ”From private conversations we know that this has occurred to the following mathematicians and discussed among themselves: T.Gelander, A.Shalev, M.Larsen, and A.Lubotzky. The question was mentioned several times in the Einstein Institute Algebra Seminar. This conjecture was independently raised in the paper N.Linial and D.Puder, ”Words maps and spectra of random graph lifts”, Random Structures and Algorithms 37 (2010), no 1, 100-135.”

In the paper of D.Puder "On primitive words II: measure preservation" the positive solution of the Conjecture 1 was given for $r = 2$.

Later the positive decision for each $r \geq 2$ was given by D.Puder and O.Porzanchevski, " Measure Preserving Words are Primitive" arXiv, 15 Feb. 2012

Theorem (Puder, Porzanchevski, 2012) An element $v \in F_r,\ r \geq 2$, is primitive iff it is measure preserving.

Theorem (Topping, 1966). An element $v \in F_2$ is primitive iff it is unimodular.

Theorem (Umirbaev, 1994). An element $v \in F_r,\ r \geq 2$, is primitive iff it is unimodular.

Let $v \in F_r,\ r \geq 2$. Thus, from results of Umirbayev, Puder, and Porzanchevski we obtain that the following conditions are equivalent:

(1) The element v is primitive;
(2) The element v is unimodular;
(3) The element v preserves measure (on the variety of all groups).

PRIMITIVE, MEASURE PRESERVING

and UNIMODULAR

SYSTEMS OF ELEMENTS

(on the variety of all groups).

Measure preserving systems of elements.

The notion of a measure preserving element can be extended to a set of elements in F_r as follows. Consider an ordered set (system) of elements $\{v_1, \dots, v_m\}$, $1 \leq m \leq r$, in F_r. Define the verbal mapping $\varphi_{\{v_1,\dots,v_m\}}$ from G^r to G^m by assigning to each $\overline{g} = (g_1, \dots, g_r) \in G^r$ the element $(v_1(\overline{g}), \dots, v_m(\overline{g})) \in G^m$:

$$\varphi_{\{v_1,\dots,v_m\}} : (g_1, \dots, g_r) \ \mapsto \ (v_1(g_1, \dots, g_r), \dots, v_m(g_1, \dots, g_r)).$$

Definition. A system of elements $\{v_1, \ldots, v_m\}$ preserves measure on G if every $\overline{g} \in G^m$ is the image of exactly $|G|^{r-m}$ elements under $\varphi_{\{v_1,\ldots,v_m\}}$, i.e. with probability $|G|^{-m}$.

Definition. A system of elements $\{v_1, \ldots, v_m\}$ that preserves measure on every finite group G is called measure preserving (on the variety of all groups) .

Definition. A system of elements $\{v_1, \ldots, v_m\}$ in a free group F_r is called primitive if it can be complemented to a basis for F_r.

Let $\overline{v} = \{v_1, \ldots, v_m\},\ 1 \leq m \leq r$, be a system of elements of the free group F_r and let

$$J(\overline{v}) = (\partial_i(v_j))_{m\times r}$$

be the Jacobi matrix.

Definition. The system of elements $\overline{v}$ is called unimodular if there is a matrix $A_{r\times m}$ over the ring $\mathbb{Z}(F_r)$ such that $J(\overline{v}) \cdot A = E_{m\times m}$ (identity matrix)

SOME KNOWN FACTS AND CONJECTURES
ABOUT PRIMITIVE, MEASURE PRESERVING,
AND UNIMODULAR SYSTEMS OF ELEMENTS
(on the variety of all groups).

The connections between primitive and unimodular systems of elements was established by U.U.Umirbaev.

Theorem (Umirbaev, 1994). A system of elements $\{v_1, \ldots, v_m\}$, $1 \leq m \leq r$, of the free group F_r, $r \geq 2$, is primitive iff it is unimodular.

Theorem (Puder, 2011, Puder, Parzanchevski, 2012). For $m = 1, r-1, r$ a system of elements $\{v_1, \ldots, v_m\}$ of the free group F_r is primitive iff it preserves measure (on the variety of all groups).

PRIMITIVE and MEASURE PRESERVING
SYSTEMS of ELEMENTS
(on a variety $\mathfrak{M}$).

Suppose we consider only groups in some variety $\mathfrak{M}$ as finite groups G. Let $V = V(\mathfrak{M})$ be the verbal subgroup in F_r corresponding to this variety. All values of $v \in V$ on G are equal to the unit. Therefore, $\{v_1, \ldots, v_m\}$ in the definition of systems of measure preserving elements on the group G may be taken from the relatively free group $F_r(\mathfrak{M}) = F_r/V$.

Definition. A system of elements $\{v_1, \ldots, v_m\}$, $1 \leq m \leq r$, in a relatively free group $F_r(\mathfrak{M})$ preserves measure on the variety $\mathfrak{M}$ if it preserves measure on every finite group $G \in \mathfrak{M}$.

Suppose that a variety $\mathfrak{M}$ is a product $\mathfrak{M} = \mathfrak{A}\mathfrak{B}$ of the variety $\mathfrak{A}$ of all abelian groups and some variety $\mathfrak{B}$. Thus,

$$\mathfrak{M} = \{ G \mid G/A \in \mathfrak{B},\ A \in \mathfrak{A} \}.$$

Then

$$F_r(\mathfrak{M}) \cong F_r/[V, V],$$

where V is the verbal subgroup of F_r corresponding to the variety $\mathfrak{B}$. Note that the Fox derivations are well defined on $F_r(\mathfrak{M})$ if there values obtained in $\mathbb{Z}(F_r/V) \cong \mathbb{Z}(F_r(\mathfrak{B}))$.
In particular the concept of an unimodular element is defined correctly for variety $\mathfrak{A}^2$ of all metabelian groups.

Definition. Let $\overline{v} = \{v_1, \ldots, v_m\}$, $1 \leq m \leq r$, be a system of elements of relatively free group $F_r(\mathfrak{A}\mathfrak{B})$ and let $J(\overline{v}) = (\partial_i(v_j))_{m\times r}$ be the Jacobi matrix over the ring $\mathbb{Z}(F_r(\mathfrak{B}))$. The system of elements $\overline{v} = \{v_1, \ldots, v_m\}$ is called unimodular (on the variety $\mathfrak{A}\mathfrak{B}$) if there is a matrix $A_{r\times m}$ over the ring $\mathbb{Z}(F_r(\mathfrak{B}))$ such that $J(\overline{v}) \cdot A = E_{m\times m}$.

For the variety of all metabelian groups the following criterion of primitiveness takes place.

THEOREM. A system of elements $\{v_1, \ldots, v_m\}$, $1 \leq m \leq r$, in the free metabelian group $F_r(\mathfrak{A}^2)$ is primitive iff it is unimodular.

Denote by $\mathcal{P}(\mathfrak{M})$ the set of all primitive systems $\{v_1, \ldots, v_m\}$ in the group $F_r(\mathfrak{M})$ and by $\mathcal{S}(\mathfrak{M})$ the set of all measure preserving systems on the variety $\mathfrak{M}$.

QUESTION 1. For which solvable varieties $\mathfrak{M}$ the sets $\mathcal{P}(\mathfrak{M})$ and $\mathcal{S}(\mathfrak{M})$ coincide?

NEW RESULTS

THEOREM 1 (Timoshenko, 2013). Let $\mathfrak{N}_c$ be the variety of all nilpotent groups of class at most c. A system of elements $\{v_1, \ldots, v_m\}$, $1 \leq m \leq r$, of the free nilpotent group $F_r(\mathfrak{N}_c)$ is primitive iff the system preserves measure on the variety $\mathfrak{N}_c$, i.e. $\mathcal{P}(\mathfrak{N}_c) = \mathcal{S}(\mathfrak{N}_c)$.

THEOREM 2 (Timoshenko, 2013). A system of elements $\{v_1, \ldots, v_m\}$, $1 \leq m \leq r$, in $F_r(\mathfrak{N}_c\mathfrak{A})$ preserves measure on the variety $\mathfrak{N}_c\mathfrak{A}$ iff the system is primitive, i.e. $\mathcal{P}(\mathfrak{N}_c\mathfrak{A}) = \mathcal{S}(\mathfrak{N}_c\mathfrak{A})$.

In particular, Theorem 2 is true for the variety $\mathfrak{A}^2$ of all metabelian groups.

For $\mathfrak{M} = \mathfrak{A}\mathfrak{B}$ denote by $\mathcal{U}(\mathfrak{M})$ the set of all unimodular systems of elements in the group $F_r(\mathfrak{M})$.

Therefore

$$\mathcal{P}(\mathfrak{A}^2) = \mathcal{U}(\mathfrak{A}^2) = \mathcal{S}(\mathfrak{A}^2).$$

Thus, the Question 1 has an affirmative answer for the varieties $\mathfrak{N}_c$, $\mathfrak{N}_c\mathfrak{A}$ and partially for the variety $\mathfrak{G}$ of all groups. Therefore a new question arises:

Question 2. Is there a variety $\mathfrak{M}$ and non-primitive element $v \in F_r(\mathfrak{M})$ such that v preserves measure on $\mathfrak{M}$?

THEOREM 3 (Timoshenko, 2013). There is some unimodular non-primitive element $v \in F_2(\mathfrak{A}\mathfrak{N}_2)$. This element preserves measure on the variety $\mathfrak{A}\mathfrak{N}_2$. We can take $v = x_1[x_1, x_2, x_2, x_1, x_2] \in F_2(\mathfrak{A}\mathfrak{N}_2)$. Thus,

$$\mathcal{P}(\mathfrak{A}\mathfrak{N}_c) \subsetneq \mathcal{S}(\mathfrak{A}\mathfrak{N}_c).$$

The proof of this theorem is based on the following criterion of primitiveness for locally finite varieties of groups.

THEOREM (Timoshenko, 1998). Let $\mathfrak{M}$ be a variety generated by some finite group, $n > 0$, $\mathfrak{A}_n$ be a variety of abelian groups, whose exponent divides n, $1 \leq m \leq r$. The elements $\{v_1, \ldots, v_m\}$ of the group $F_r(\mathfrak{A}_n\mathfrak{M})$ form a primitive system iff
(1) the system of elements $\{v_1, \ldots, v_m\}$ is unimodular over the ring $\mathbb{Z}_n(F_r(\mathfrak{M}))$;
(2) the images of $\{v_1, \ldots, v_m\}$ in the group $F_r(\mathfrak{M})$ form a primitive system.

Suppose that $v_1, v_2 \in F_r$ belong to the same orbit of the action of $Aut(F_r)$. This clearly implies that they induce the same measure on every finite group.

Conjecture 2. The converse is also true.

This Conjecture was formulated by D.Puder in his paper "On primitive words II: measure preservation".

I don't know any results on this Conjecture (in variety of all groups)

We are interested in this Conjecture for a variety of solvable groups.

Question 3. Is there a solvable variety $\mathfrak{M}$ and two elements v_1 and v_2 in $F_r(\mathfrak{M})$ such that they belong to the different orbits of the action of $Aut(F_r(\mathfrak{M}))$ but induce the same measure on the variety $\mathfrak{M}$?

COROLLARY. There are two elements $v_1 = x_1$ and $v_2 = x_1[x_1, x_2, x_2, x_1, x_2]$ in $F_2(\mathfrak{A}\mathfrak{N}_2)$ such that they belong to the different orbits of the action of $Aut(F_2(\mathfrak{A}\mathfrak{N}_2))$ but they induce the same measure on the variety $\mathfrak{A}\mathfrak{N}_2$.

PROFINITE METABELIAN GROUPS

We will give other definition of measure preserving system of elements. New definition is equivalent to the previous one for the abstract (not topological) groups. But it can be applied also to topological groups.
Fix some finite group $G \in \mathfrak{M}$ and select a homomorphism $\alpha \in Hom(F_r(\mathfrak{M}), G)$ uniformly at random. A homomorphism from a relatively free group is uniquely determined by choosing the images of the elements of a basis, so that every homomorphism is chosen with probability $1/|G|^r$.

Definition. We say that $\{v_1, \ldots, v_m\}$ is measure preserving if for every finite group $G \in \mathfrak{M}$ and randomly chosen homomorphism $\alpha \in Hom(F_r(\mathfrak{M}), G)$ the m-tuple $(\alpha(v_1), \ldots, \alpha(v_m))$ is uniformly distributed in G^m.

Let $\widehat{F}_r$ be the profinite completion of F_r and F_r is naturally embeded in $\widehat{F}_r$. Every basis of F_r is then a basis for $\widehat{F}_r$, so a primitive word $v \in F_r$ is also primitive as an element of $\widehat{F}_r$. It is conjectured that the converse also holds:

Conjecture 3. A word $v \in F_r$ is primitive in $\widehat{F}_r$ iff it is primitive in F_r.

Equivalent formulation.

Conjecture 3. The primitive elements of free group F_r form a closed set in the profinite topology of F_r.

The positive decision of Conjecture 3 was given by C.Mery.

Theorem. $v \in F_r$ is primitive as an element of $\widehat{F}_r$ iff it is primitive in F_r.

The similar question arises to varieties of groups. We are interested in this question for the variety $\mathfrak{A}^2$ of all metabelian groups.

QUESTION 4. Let S be a free metabelian group of rank r and $\widehat{S}$ its profinite completion. Do the primitive elements of S form a closed set in the profinite topology of the group S?

THEOREM 4 (Timoshenko, 2013). Let $\widehat{S}$ be the free profinite metabelian group of rank r. A system of elements $\{v_1, \ldots, v_m\} \in \widehat{S}, 1 \le m \le r$, is primitive iff this system preserves measure on the variety of all profinite metabelian groups.

THEOREM 5 (Timoshenko, 2013). A system of elements $\{v_1, \ldots, v_m\}$, $1 \le m \le r$, of the free metabelian group S of rank r is primitive iff it is primitive in $\widehat{S}$.

In particular the primitive elements of free metabelian group S form a closed set in profinite topology of S.

[1] Timoshenko E.I., Primitive and measure preserving systems of elements on the varieties of metabelian and metabelian profinite groups, SMZ, Vol. 54, No 1, pp 153-159, 2013.

[2] Timoshenko E.I., Systems of elements preserving measure on the varieties of groups, Math. Sbornik, 2013.

Thanks for attention!