DIALECTICAL SYSTEMS AND QUASI-DIALECTICAL SYSTEMS: TWO APPROACHES TO TRIAL-AND-ERROR MATHEMATICS

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Dialectical systems were introduced by Roberto Magari in order to describe a trial-and-error approach to mathematical theories, in which axioms may be rejected or accepted through a trial-and-error process: the *final theses*, i.e. the axioms that are eventually accepted, form a Δ^0_2 set. The *dialectical sets* are the sets that consist of the final theses of some dialectical system. When equipped with the deductive machinery of a consistent formal theory T, a dialectical system produces a consistent Δ^0_2 completion of T, so that dialectical systems describe "deductive procedures" that in some sense escape the incompleteness phenomenon; it is also possible to come up with dialectical systems that, among their final theses, have a statement naturally expressing their consistency. The basic ingredient of a dialectical system is that when a proposed axiom seems to produce a contradiction, then we "discard it", and we try with next one. We enrich this mechanism with a "revision" procedure *à la* Lakatos: when an axiom seems inadequate, even if not contradictory, then we replace it with a new one. The enriched systems are called *quasi-dialectical systems*, and the corresponding sets of *final theses* are called *quasi-dialectical sets*, forming a class that extends the class of dialectical sets. We prove that quasi-dialectical sets lie in the same Turing degrees as dialectical sets, namely the c.e. Turing degrees; also the two classes provide the same enumeration degrees, namely the Π^0_1 enumeration degrees. So, in some sense, dialectical systems and quasi-dialectical systems display the same information content. Nevertheless, the class of quasi-dialectical sets is much larger than the class of dialectical sets: every dialectical set can be shown to be ω-c.e., whereas quasi-dialectical sets spread out through all classes of the Ershov hierarchy of Δ^0_2 sets: for every ordinal notation $a \in \mathcal{O}$, with $|a|_{\mathcal{O}} > 0$, there is a quasi-dialectical set $A \in \Sigma^{-1}_a \smallsetminus \bigcup_{b <_{\mathcal{O}} a} \Sigma^{-1}_b$. As to dialectical sets, we show that for every finite $n > 1$ there exist n-c.e. dialectical sets, which are not $(n-1)$-c.e. (the statement is not true of $n = 1$, as already observed by Magari); and there exist ω-c.e. dialectical sets which are not n-c.e., for every finite n.

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