Heyting-Ockham Logic and Hyperintentionality. (From ‘Star’ and ‘Perp’ to HYPE via WFS.)

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Outline

- H. Leitgeb, 'HYPE: A System of Hyperintentional logic (with an Application to Semantical Paradoxes), JPhL, online 2018.
- M.J. Dunn, 'Star and Perp: Two Treatments of Negation', Phil. Persp., 1993.
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  where N^* is a Heyting-Ockham logic of [COP2006].

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HYPE: the logic of hyperintensional contexts

"It is well known that it seems possible to have a situation in which there are two propositions p and q which are logically equivalent and yet are such that a person may believe the one and not the other. If we regard a proposition as a set of possible worlds then two logically equivalent propositions will be identical, and so if 'x believes that' is a genuine sentential functor, the situation described in the opening sentence could not arize. I call this the paradox of hyperintentional contexts. Hyperintensional contexts are simply contexts which do not respect logical equivalence."

[M.J. Cresswell, Hyperintensional logic (1974)]

"...we may have a person so logically blind that he may believe p without believing $\sim\sim p$"

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HYPE: the logic of hyperintensional contexts

"The twenty-first century is seeing a hyperintensional revolution."

[D. Nolan, Hyperintensional metaphysics (2014)]

"If the hyperintensional revolution is coming at all, HYPE might be one of its future venues. Alternatively, HYPE might serve as a laboratory setting in which the promise of the hyperintensional revolution gets disconfirmed by experiment."

[H. Leitgeb, HYPE: A system of hyperintensional . . . (2018)]

HYPE: syntax

- Propositional language $\{\vee, \wedge, \to, \neg, \mathbf{t}\}$ over
$$\mathrm{Prop} = \{p_1, p_2, \ldots\}.$$
- $\mathrm{Lit} = \{p, \neg p \mid p \in \mathrm{Prop}\}$.
- Define $\overline{(\cdot)} : \mathrm{Lit} \to \mathrm{Lit}$ as follows:
$$\overline{p} := \neg p \quad \text{and} \quad \overline{\neg p} := p.$$
- $\varphi \leftrightarrow \psi := (\varphi \to \psi) \wedge (\psi \to \varphi)$, $\quad \mathbf{f} := \neg \mathbf{t}$

HYPE: models

Model $\mathfrak{M} = \langle S; V; \circ; \perp \rangle$ is such that:

- S is a non-empty set (the set of states).
- $V : S \to 2^{\mathrm{Lit}}$ (the valuation function).
- $\circ$ is a partial function from S^2 to S (the fusion function) such that:
  - if $(s \circ s') \downarrow$, then $V(s \circ s') \supseteq V(s) \cup V(s')$;
  - $(s \circ s) \downarrow$ and $s \circ s = s$;
  - if $(s \circ s') \downarrow$, then $(s' \circ s) \downarrow$ and $s \circ s' = s' \circ s$;
  - if $(s \circ s') \circ s''$ is defined, then $s \circ ((s \circ s') \circ s'')$ is defined, and $s \circ ((s \circ s') \circ s'') = (s \circ s') \circ s''$.

HYPE: models, incompatibility relation

Model $\mathfrak{M} = \langle S; V; \circ; \perp \rangle$ is such that:

- $\perp$ is a binary symmetric relation on S (the incompatibility relation), s.t.:
  - if there is an α with $\alpha \in V(s)$ and $\overline{\alpha} \in V(s')$, then $s \perp s'$;
  - If $s \perp s'$ and both $s \circ s''$ and $s' \circ s'''$ are defined, then $s \circ s'' \perp s' \circ s'''$.
- For every $s \in S$ there is a unique $s^* \in S$, such that:
  - $V(s^*) = \{\overline{\alpha} \mid \alpha \notin V(s)\}$;
  - $s^{**} = s$;
  - s and s^* are not incompatible with each other: $s \not\perp s^*$;
  - s^* is largest having the last property: if $s \not\perp s'$, then $s' \circ s^* = s^*$.

First Remarks on HYPE-models

- [Information order]
  Let $s \leq s'$ iff $(s \circ s') \downarrow$ and $s \circ s' = s'$.
  Then $\leq$ is a partial order on S.

- [Antimonotonicity] If $s \leq s'$, then $s'^{*} \leq s^{*}$.

HYPE: Satisfaction of Formulas on States

- $s \models \alpha$ iff $\alpha \in V(s)$, where $\alpha \in \mathrm{Lit}$.
- $s \models \neg\varphi$ iff for all s' (if $s' \models \varphi$, then $s \perp s'$).
- $s \models \varphi \wedge \psi$ iff $s \models \varphi$ and $s \models \psi$.
- $s \models \varphi \vee \psi$ iff $s \models \varphi$ or $s \models \psi$.
- $s \models \varphi \rightarrow \psi$ iff for all s' (if $s' \models \varphi$ and $(s \circ s')\downarrow$, then $s \circ s' \models \psi$).
- $s \models \mathbf{t}$.

Consequence relation and theorems

- s is world-like if it has no gaps and gluts in the following sense:

$$\forall\varphi(s \models \varphi \;\Leftrightarrow s \not\models \neg\varphi)$$

- $\{s \mid s \models \varphi\}$ is a hyperintensional of φ

- Hyperintensional gaps and gluts should imply (S.O.) the fail of

$$\forall\varphi(s \models \varphi \;\Leftrightarrow not(s \not\models \varphi))$$

- $\varphi_1, \ldots, \varphi_n \models \psi$ iff $\forall\mathfrak{M}\ \forall s \in S$ (if $s \models \varphi_1, \ldots s \models \varphi_n$, then $s \models \psi$)

- $\models \psi$ iff $\forall\mathfrak{M}\ \forall s \in S\ (s \models \psi)$

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HYPE: First Remarks on Satisfaction and Consequence

- [Monotonicity]

  If $s \models \varphi$ and $s \leq s'$, then $s' \models \varphi$.

- [Superintuitionistic Implication]

  $s \models \varphi \to \psi$ iff for all s' (if $s \leq s'$ and $s' \models \varphi$, then $s' \models \psi$)

- [Star Negation]

  $s \models \neg\varphi$ iff $s^* \not\models \varphi$. (Recall $V(s^*) = \{\overline{\alpha} \mid \alpha \notin V(s)\}$)

- [FDE-extension]

  FDE coincides with $\{\to, \mathbf{t}\}$-free fragment of one-premiss $\models$

Hilbert Style Calculus for HYPE

- $\vdash \mathbf{t}$
- $\vdash \varphi \to (\psi \to \varphi)$
- $\vdash (\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$
- $\vdash (\varphi \wedge \psi) \to \varphi \quad \vdash (\varphi \wedge \psi) \to \psi$
- $\vdash \varphi \to (\varphi \vee \psi) \quad \vdash \psi \to (\varphi \vee \psi)$
- $\vdash \varphi \to (\psi \to (\varphi \wedge \psi))$
- $\vdash (\varphi \to \chi) \to ((\psi \to \chi) \to ((\varphi \vee \psi) \to \chi))$
- $\vdash \neg\neg\varphi \leftrightarrow \varphi$
- $\vdash \neg(\varphi \wedge \psi) \leftrightarrow (\neg\varphi \vee \neg\psi)$
- $\vdash \neg(\varphi \vee \psi) \leftrightarrow (\neg\varphi \wedge \neg\psi)$

- $\vdash \varphi \to \psi / \vdash \neg\psi \to \neg\varphi$
- $\varphi, \varphi \to \psi \vdash \psi$

Strong Completeness and Other Results [Leitgeb 2018]

- $\vdash$ satisfies Deduction Theorem.
- [Strong Completeness] $\Gamma \vdash \psi$ iff $\Gamma \models \psi$.
- HYPE has the finite model property.
- The set of logical truths of HYPE is decidable.
- HYPE has the Disjunction Property: if $\models \varphi \vee \psi$, then $\models \varphi$ or $\models \varphi$.

Two treatments of negation by [Dunn 93]

$$(\neg *) \quad x \models \neg\varphi \quad \text{iff} \quad x^* \not\models \varphi$$

$$(\neg \perp) \quad x \models \neg\varphi \quad \text{iff} \quad \forall y (y \models \varphi \;\Rightarrow\; y \perp x)$$

Two treatments of negation by [Dunn 93]

- The origins of $(\neg *)$ are in the representation theorem for De Morgan lattices (quasi-Boolean algebras) by [Białynicki-Birula & Rasiowa 57].
- [R. Routley & V. Routley, *Semantics of First-Degree Entailment* 72]
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Two treatments of negation by [Dunn 93]

- "…it is convenient to suppose that we are dealing with so called "UCLA propositions", i.e., sets of verifying states. …This allows us to talk of negation directly as an operation on sets, and omits the "middle man" of syntax."
- An information frame is a structure $(U, \sqsubseteq)$, where $\sqsubseteq$ is a partial order on U.
- A star frame is an information frame $(U, \sqsubseteq, *)$ with function $* : U \rightarrow U$ antitone w.r.t. $\sqsubseteq$.
- A perp frame is an information frame $(U, \sqsubseteq, \perp)$ with $\perp \subseteq U \times U$ isotone w.r.t. $\sqsubseteq$ in each of its arguments. $xCy := not(x \perp y)$.

Two treatments of negation by [Dunn 93]

- A star-crossed perp frame is a perp frame $(U, \sqsubseteq, \perp)$ such that for each $x \in U$, $C_x = \{y \mid yCx\}$ has a greatest element x^*
- The propositions $P(U)$ on an information frame $(U, \sqsubseteq)$ are subsets of U upwards closed w.r.t. $\sqsubseteq$.
- $A^{\perp} = \{x \mid A \perp x\} = \{x \mid \forall y \in A(y \perp x)\}$
- $A^{\sim} = \{x \mid x^* \notin A\}$

Two treatments of negation by [Dunn 93]

Theorem (Basic theorem. [Dunn 93])

(i) *Given a star frame, a (star-crossed) perp frame can be obtained with $x \perp y$ defined as not$(x \sqsubseteq y^*)$, and with the result that for any proposition A, $A^{\perp} = A^{\sim}$.*

(ii) *And given a star-crossed perp frame, a star frame can be obtained with x^* defined as the greatest element of C_x, and again the result is that for propositions $A^{\perp} = A^{\sim}$.*

From incompatibility to compatibility

$$(\neg\perp) \quad x \models \neg\varphi \quad \text{iff} \quad \forall y (y \models \varphi \;\Rightarrow\; y \perp x)$$

- Rewording of $(\neg\perp)$ via compatibility relation

$$x \models \neg\varphi \text{ iff } \forall y (y\mathsf{C}x \;\Rightarrow\; y \not\models \varphi),$$

where $x\mathsf{C}y := \mathit{not}(x \perp y)$.

Vakarelov's Theory of Negation

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- Only distributive logics are considered.

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Distributive logics

- Logic is a set of sequents $\varphi \vdash \psi$ containing axioms:

$$\varphi \vdash \varphi, \quad \mathbf{f} \vdash \varphi, \quad \varphi \vdash \mathbf{t},$$

$$\varphi \wedge \psi \vdash \varphi, \psi, \quad \varphi, \psi \vdash \varphi \vee \psi$$

- and closed under substitution and the following rules:

$$\frac{\varphi \vdash \psi, \ \psi \vdash \chi}{\varphi \vdash \chi}, \quad \frac{\chi \vdash \varphi, \ \chi \vdash \psi}{\chi \vdash \varphi \wedge \psi}, \quad \frac{\varphi \vdash \chi, \ \psi \vdash \chi}{\varphi \vee \psi \vdash \chi}$$

- Logic containing all sequents $\varphi \wedge (\psi \vee \chi) \vdash (\varphi \wedge \psi) \vee (\varphi \wedge \chi)$ is distributive.

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Theories, Co-theories, Prime Theories

- A set x of formulas is a theory in L iff
  - (i) $\mathbf{t} \in x$
  - (ii) if $\varphi \in x$ and $\varphi \vdash \psi \in L$, then $\psi \in x$.
  - (iii) if $\varphi \in x$ and $\psi \in x$, then $\varphi \wedge \psi \in x$.
- A set x of formulas is a co-theory in L iff
  - (i) $\mathbf{f} \notin x$
  - (ii) if $\varphi \in x$ and $\varphi \vdash \psi \in L$, then $\psi \in x$.
  - (iii) if $\varphi \vee \psi \in x$, then $\varphi \in x$ or $\psi \in x$.
- x is a prime theory in L if x is a theory in L and x is a co-theory in L.
- $\mathrm{P}(L)$ is the set of all prime theories in L.

Characterization of Distributive Logics

- *Interpolation Lemma.* If x is a theory in L, z is a co-theory in L, and $x \subseteq z$, then there is an $y \in \mathrm{P}(L)$ s.t. $x \subseteq y \subseteq z$.
- *Extension Lemma.* If x is a theory in L and $\varphi \notin x$, then there is an $y \in \mathrm{P}(L)$ s.t. $x \subseteq y$ and $\varphi \notin y$.
- *Co-extension Lemma.* If z is a co-theory in L and $\varphi \in z$, then there is an $y \in \mathrm{P}(L)$ s.t. $\varphi \in y \subseteq z$.
- *Separation Lemma.* If $\varphi \vdash \psi \notin L$, then there is an $y \in \mathrm{P}(L)$ s.t. $\varphi \in x$ and $\psi \notin x$.

Characterization of Distributive Logics

Theorem

The following conditions are equivalent for any logic L:

(i) *L is distributive;*

(ii) *Interpolation Lemma holds for L.*

(iii) *Extension Lemma holds for L.*

(iv) *Co-extension Lemma holds for L.*

(v) *Separation Lemma holds for L.*

Characterization of Intuitionistic and Classical Logics

- **Int** is the least distributive logic closed under

$$\frac{\chi \wedge \varphi \vdash \psi}{\chi \vdash \varphi \to \psi} \quad \text{and} \quad \frac{\chi \vdash \varphi \to \psi}{\chi \wedge \varphi \vdash \psi}$$

- **Int** $\subseteq L$ iff

$$\varphi \to \psi \in x \quad \Leftrightarrow \quad \forall y \in \mathrm{P}(L)((x \subseteq y \;\&\; \varphi \in y) \;\Rightarrow\; \psi \in y)$$

- **CL** is the least distributive logic containing **Int** and all sequents $\mathbf{t} \vdash \varphi \vee (\varphi \to \psi)$

- **CL** $\subseteq L$ iff $\forall x \in \mathrm{P}(L)(\varphi \to \psi \in x \;\Leftrightarrow\; (\varphi \notin x \vee \psi \in x))$

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- Theory x is consistent if

$$\forall \varphi \, (\neg \varphi \in x \;\Rightarrow\; \varphi \notin x)$$

- Theories x and y are compatible, xRy, if

$$\forall \varphi \, (\neg \varphi \in x \;\Rightarrow\; \varphi \notin y)$$

- For a set x of formulas, put

$$g(x) = \{\varphi \mid \neg \varphi \notin x\}$$

- Then xRy iff $y \subseteq g(x)$.

Regular logics

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Regular logics

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Regular logics

- Theory x is normal if $\neg\varphi \in x$ for some φ
- $\mathrm{N}(L)$ is the set of all normal theories in L
- Distributive logic L is regular if for all φ and $x \in \mathrm{P}(L)$,

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Characterization of regular logics

Let L be a distributive logic with negation in their language. The following conditions are equivalent:

- L is a regular logic;
- if x is a normal theory in L, then $g(x)$ is a co-theory in L.
- For any φ and ψ the following conditions hold:

Reg$_0$ if $\varphi \vdash \psi \in L$, then $\neg\psi \vdash \neg\varphi \in L$

Reg $\neg\varphi \wedge \neg\psi \vdash \neg(\varphi \vee \psi) \in L$

Characterization of normal logics

- Regular logic L is normal iff all its prime theories are normal.

- L is normal iff $\mathbf{t} \vdash \neg\mathbf{f} \in L$ iff

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- The logic N from [Dosen, *Negation as a modal operator*, 1986] is $\{\mathbf{t}, \mathbf{f}\}$-free fragment of the least normal logic with intuitionistic implication.

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Standard logics

- Regular logic L is standard if:

  A1 $\varphi \wedge \neg(\varphi \wedge \psi) \vdash \neg\psi \in \boldsymbol{L}$

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- Vakarelov's subminimal logic $(\neg\varphi = (\varphi \to \neg\mathbf{t}) \wedge \neg\mathbf{f})$ is the least standard logic with intuitionistic implication

- Johansson's minimal logic $(\neg\varphi = \varphi \to \neg\mathbf{t})$ is the least normal standard logic with intuitionistic implication

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Co-regular logics

- Recall that

$$g(x) = \{\varphi \mid \neg\varphi \notin x\}$$

- Theory x is complete iff

$$\forall\varphi\ (\neg\varphi \notin x \;\Rightarrow\; \varphi \in x) \text{ iff } g(x) \subseteq x$$

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Characterization of co-regular logics

Let L be a distributive logic with negation in their language. The following conditions are equivalent:

- L is a co-regular logic;
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- Co-regular logic L is co-normal iff there are no co-normal prime theories in L.
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- The logic $CC_\omega = C_\omega + \{\frac{\varphi\to\psi}{\neg\psi\to\neg\varphi}\}$ from [R. Sylvan 1990] is $\{\mathbf{t},\mathbf{f}\}$-free fragment of a co-normal logic with intuitionistic implication and reflexive and transitive completion relation.

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Completion vs Compatibility

- Let L be a co-regular logic. For any $x, y \in \mathrm{P}(L)$:
  - xRy iff $x \notin \mathrm{N}'(L)$ & $\forall z \in \mathrm{P}(L)(xR'z \Rightarrow y \subseteq z)$
  - $x \in \mathrm{N}(L)$ iff ($x \in \mathrm{N}'(L)$ or $\exists z \in \mathrm{P}(L)(xR'z)$)
- Let L be a regular logic. For any $x, y \in \mathrm{P}(L)$:
  - $xR'y$ iff $x \notin \mathrm{N}(L)$ & $\forall z \in \mathrm{P}(L)(xRz \Rightarrow z \subseteq y)$
  - $x \in \mathrm{N}'(L)$ iff ($x \in \mathrm{N}(L)$ & $\forall z \in \mathrm{P}(L)$ not(xRz))

Completion plus Compatibility

- Let L be a normal and co-normal logic. For any $x, y \in \mathrm{P}(L)$:
  - $g(x) \in \mathrm{P}(L)$
  - Since xRy iff $y \subseteq g(x)$

$$\forall y \in \mathrm{P}(L)(xRy \;\Rightarrow\; \varphi \notin y) \;\text{ iff }\; \varphi \notin g(x)$$

  - Since $xR'y$ iff $g(x) \subseteq y$

$$\exists y \in \mathrm{P}(L)(xR'y \;\&\; \varphi \notin y) \text{ iff } \varphi \notin g(x)$$

  - In any case,

$$\neg\varphi \in x \text{ iff } \varphi \notin g(x)$$

Logics with Star Negation

- Heyting-Ockham logic N^* from [Cabalar, Odintsov, Peirce 2006] is the least normal and co-normal logic with intuitionistic implication. (N^* is the logic with $\to$ of star frames $(W, \sqsubseteq, *)$)
- FDE is $\{\mathbf{t}, \mathbf{f}\}$-free fragment of the least normal and co-normal logic containing all sequents $\varphi \dashv\vdash \neg\neg\varphi$.
(FDE is the logic without $\to$ of star frames $(W, \sqsubseteq, *)$ with $x = x^{**}$)
- HYPE is the least normal and co-normal logic with intuitionistic implication containing all sequents $\varphi \dashv\vdash \neg\neg\varphi$.
(HYPE is the logic with $\to$ of star frames $(W, \sqsubseteq, *)$ with $x = x^{**}$)
- HYPE $= N^* + \{\varphi \leftrightarrow \neg\neg\varphi\}$
- DP counterexample of S. Drobyshevich for N^* and for HYPE

$$(q \to \neg p) \vee (q \to \neg(p \to \mathbf{f})) \in N^*(\text{HYPE})$$

Logic programs

1. *At* is a propositional signature.
2. Logic program Π is a set of rules of the form

$$(r) \quad p_1 \vee \ldots \vee p_k \leftarrow q_1 \wedge \ldots \wedge q_n \wedge \neg q_{n+1} \wedge \ldots \wedge \neg q_{n+m},$$

   where $p_i, q_j \in At$.
3. Π is normal if $k = 1$ for all rules in Π.
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Partial models

- Partial model $\mathcal{M}$ is a consistent set of literals,

  i.e., $\{p, \neg p\} \not\subseteq \mathcal{M}$ for any p.

p is true at $\mathcal{M}$	iff	$p \in \mathcal{M}$
p is false at $\mathcal{M}$	iff	$\neg p \in \mathcal{M}$
p is undefined at $\mathcal{M}$	iff	$p \notin \mathcal{M}$, $\neg p \notin \mathcal{M}$

- Alternatively, $\mathcal{M} = (M, M')$, where $M \subseteq M' \subseteq At$.

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Truth and Information Orderings

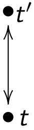
Orderings on partial models: truth $\leq$ and information $\preceq$.

Let $\mathcal{M}_1 = (M_1, M_1')$ and $\mathcal{M}_2 = (M_2, M_2')$

$$\begin{array}{lcccc} \mathcal{M}_1 \leq \mathcal{M}_2 & \iff & M_1 \subseteq M_2 & \text{and} & M_1' \subseteq M_2' \\ \mathcal{M}_1 \preceq \mathcal{M}_2 & \iff & M_1 \subseteq M_2 & \text{and} & M_2' \subseteq M_1' \end{array}$$

Logic HT^*

- HT^* is given by star frame $\mathcal{W}^{HT^*} = \langle W^{HT^*}, \leq, * \rangle$, where $W^{HT^*} = \{t, t'\}$, $t \leq t'$, $t^* = t'$, and $t'^* = t$.



- Model over $\mathcal{W}^{HT^*}$ is a pair (T, T'), $T \subseteq T' \subseteq At$.

$$t \models p \;\Leftrightarrow\; p \in T \quad \text{and} \quad t' \models p \;\Leftrightarrow\; p \in T'$$

- $HT^* = N^* +$

$$\{\alpha \vee (\alpha \to \beta) \vee -\beta,\; \alpha \leftrightarrow \neg\neg\alpha,\; \alpha \wedge \neg\alpha \to \neg\beta \vee \neg\neg\beta\},$$

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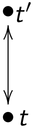

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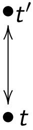

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Partial stable models

- Partial stable models [Przymusinski 94]
  $\mathcal{M}$ p-stable model $\iff$ $\leq$-minimal HT^*-model of the reduct $\Pi^{\mathcal{M}}$
- 3-valued reduct of Π w.r.t. $\mathcal{M} = (M, M')$, $M \subseteq M'$.

$\mathcal{M}$		reduct
$p \in M$	true	$\neg p \mapsto \bot$
$p \in M' \setminus M$	undefined	$\neg p \mapsto u$
$p \notin M'$	false	$\neg p \mapsto \top$

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  Well founded model [Van Gelder, Ross, Schlipf 91]
  $=$ p-stable model, minimal wrt the information ordering $\preceq$.

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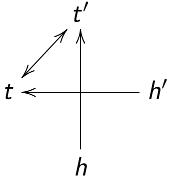

  where "higher" means $\leq$-greater
  and the arrow represents the action of $*$

- Let H, H', T, T' denote sets of atoms verified at h, h', t, t'.

- Represent a model as a pair $\langle \mathbf{H}, \mathbf{T} \rangle$, where $\mathbf{H} = (H, H')$ and $\mathbf{T} = (T, T')$.

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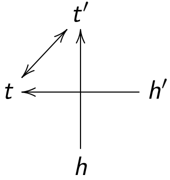

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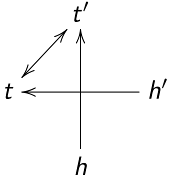

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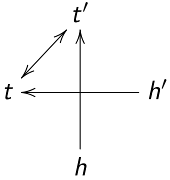
The axioms of HT^2

HT^2 equals to the extension of N^* via the axioms:



A1. $-\alpha \vee (\alpha \to (\beta \vee (\beta \to (\gamma \vee -\gamma))))$

A2. $\alpha \to \neg\neg\alpha$

A3. $\alpha \wedge \neg\alpha \to \neg\beta \vee \neg\neg\beta$

A4. $\alpha \wedge \neg\alpha \to \beta \vee (\beta \to \gamma) \vee -\gamma$

A5. $\neg\neg(\beta \vee (\beta \to \gamma) \vee -\gamma)$

A6. $\neg\neg\alpha \wedge \neg\neg\beta \to (\alpha \to \beta) \vee (\beta \to \alpha)$

Recall that $-\varphi = \varphi \to \mathbf{f}$

Partial equilibrium models

- Define an order $\trianglelefteq$ among HT^2-models as follows: $\langle \mathbf{H}_1, \mathbf{T}_1 \rangle \trianglelefteq \langle \mathbf{H}_2, \mathbf{T}_2 \rangle$ if: (i) $\mathbf{T}_1 = \mathbf{T}_2$; (ii) $\mathbf{H}_1 \leq \mathbf{H}_2$.
- $\langle \mathbf{H}, \mathbf{T} \rangle$ is said to be *total* if $\mathbf{H} = \mathbf{T}$.

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A model $\mathcal{M}$ of theory Π is a *partial equilibrium model* of Π if it is total and $\trianglelefteq$-minimal.

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Theorem

For a normal or disjunctive logic program Π, $\langle \mathbf{T}, \mathbf{T} \rangle$ is a partial equilibrium model of Π iff $\mathbf{T}$ is a partial stable model of Π.

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   $\mathcal{PEL}(\Pi)$ is the set of all *p*-equilibrium models of Π.

Definition (PEL-entailment)

$\Pi \mid\!\sim_{pel} \varphi$ if $\mathcal{M} \models_{HT^2} \varphi$ for every $\mathcal{M} \in \mathcal{PEL}(\Pi)$.

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Deductive base [Makinson 94, Dietrich 95]

Definition

Let $\mid\!\sim$ be any non-monotonic inference relation and L be a logic with monotonic inference relation $\vdash_L$. We say that a logic L is a deductive base for $\mid\!\sim$ iff

(i) $\vdash_L \subseteq \mid\!\sim$;

(ii) if $\Pi \mid\!\sim \varphi$ and $\varphi \vdash_L \psi$, then $\Pi \mid\!\sim \psi$;

(iii) if $\Pi_1 \equiv_L \Pi_2$, then $\Pi_1 \approx \Pi_2$, i.e., $\Pi_1 \mid\!\sim \varphi \Leftrightarrow \Pi_2 \mid\!\sim \varphi$ for any φ.

The deductive base L is strong if it satisfies:

$$\Pi_1 \equiv_L \Pi_2 \quad \Leftrightarrow \quad \text{for all } \Gamma \quad (\Pi_1 \cup \Gamma \approx \Pi_2 \cup \Gamma).$$

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Deductive bases for EL- and PEL-entailment

Theorem ([Cabalar, Odintsov, Pierce 06])

HT^2 is a strong deductive base for $\mid\!\sim_{pel}$.

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HT^2 is a maximal deductive base of $\mid\!\sim_{pel}$ in the lattice of N^*-extensions.

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Stages of proof

- To describe equivalent algebraic semantic for N^*-extensions: Heyting-Ockham algebras (HOc-algebras)
- To establish duality between HOc-algebras and Routley frames
- To describe subdirectly irreducible (s.i.) HOc-algebras
- To calculate $\mathrm{HS}(Alg(\mathcal{W}^{HT^2}))$ and distinguish s.i. algebras in this set
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Heyting-Ockham algebras

Definition

$\mathcal{A} = \langle A, \wedge, \vee, \rightarrow, \neg, 0, 1 \rangle$ is a *Heyting-Ockham algebra (HOc-algebra)* if the following holds:

1. The $\neg$-free reduct of $\mathcal{A}$, $\mathcal{A}^H := \langle A, \wedge, \vee, \rightarrow, 0, 1 \rangle$, is a Heyting algebra.
2. The $\rightarrow$-free reduct of $\mathcal{A}$, $\mathcal{A}^{Oc} := \langle A, \wedge, \vee, \neg, 0, 1 \rangle$, is an Ockham lattice, i.e. it is a bounded distributive lattice satisfying the identities:

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- Name "Ockham lattice" was suggested by A. Urquhart (SL, 1979)
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Relevant algebra $\langle L, \vee, \wedge, \circ, \to, \neg, 1, \top, \bot \rangle$ [Urquhart 96] satisfies the conditions:

- $\langle L, \vee, \wedge, \top, \bot \rangle$ is a bounded distributive lattice;
- $a \circ (b \vee c) = (a \circ b) \vee (a \circ c)$, $(b \vee c) \circ a = (b \circ a) \vee (c \circ a)$;
- $\neg(a \vee b) = \neg a \wedge \neg b$, $\neg(a \wedge b) = \neg a \vee \neg b$;
- $\neg\top = \bot$ and $\neg\bot = \top$;
- $a \circ \bot = \bot \circ a = \bot$;
- $1 \circ a = a$;
- $a \circ b \leq c$ iff $a \leq b \to c$.

$\circ = \wedge$, $1 = \top$, and $0 = \bot$: Relevant algebra $\Rightarrow$ HOc-algebra.

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Congruences and Filters

Definition

A non-empty $F \subseteq \mathcal{A}$ is a $*$-filter on $\mathcal{A}$ if:

1) $a \wedge b \in F$ for $a, b \in F$;
2) $b \in F$ for $a \in F$ and $a \leq b$;
3) $\neg b \rightarrow \neg a \in F$ for $a \rightarrow b \in F$.

- 1–1-Correspondence between filters and congruences:

$$\theta_F := \{(a, b) \mid a \leftrightarrow b \in F\} \text{ and } F_\theta := \{a \mid (a, 1) \in \theta\}.$$

- F is $*$-filter on $\mathcal{A}$ iff F is a filter on $\mathcal{A}^H$ and
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Necessity operator "$-\neg$"

- $-\neg 1 = 1$;
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- For any HO-algebra $\mathcal{A} = \langle A, \vee, \wedge, \rightarrow, \neg, 0, 1 \rangle$, define

$$\mathcal{A}^{\Box} := \langle A, \vee, \wedge, \rightarrow, -\neg, 0, 1 \rangle$$

- $\mathcal{A}^{\Box}$ is a $\Box$-algebra in the sense of [Wolter 97].

Theorem

The congruence lattices of an HOc-algebra $\mathcal{A}$ and of the $\Box$-algebra $\mathcal{A}^{\Box}$ coincide.
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Star frames $\rightleftarrows$ *HOc*-algebras

For a star frame $\mathcal{W} = \langle W, \leq, * \rangle$, define its algebra of cones

$$\mathcal{A}(\mathcal{W}) := \langle \langle W, \leq \rangle^{+}, \cap, \cup, \Rightarrow, \neg, \varnothing, W \rangle,$$

where

- $\langle W, \leq \rangle^{+}$ is the set of cones of $\langle W, \leq \rangle$;
- $\cap$ and $\cup$ are the intersection and the sum of sets;
- $X \Rightarrow Y := \{w \in W \mid \forall u \geq w (u \in X \text{ implies } u \in Y)\}$
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For an *HOc*-algebra $\mathcal{A}$, put

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Theorem

$a \mapsto \{F \in W(\mathcal{A}) \mid a \in F\}$, $a \in \mathcal{A}$, is an embedding of $\mathcal{A}$ into $\mathcal{A}(\mathcal{W}^{\mathcal{A}})$. If $\mathcal{A}$ is finite, then it is an isomorphism.

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- $F^* := \{a \mid \neg a \notin F\}$.

Theorem

$a \mapsto \{F \in W(\mathcal{A}) \mid a \in F\}$, $a \in \mathcal{A}$, is an embedding of $\mathcal{A}$ into $\mathcal{A}(\mathcal{W}^{\mathcal{A}})$.

If $\mathcal{A}$ is finite, then it is an isomorphism.

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- Information frames with modified propositions.

  The 2-propositions $P^2(U)$ on an information frame $(U, \sqsubseteq)$ are pairs (A, B) of upwards closed w.r.t. $\sqsubseteq$ subsets of U.

- Equivalently, $(U, \sqsubseteq, \models^+, \models^-)$

- Different ways to define implication

  - $x \models^- \varphi \to \psi \;\Leftrightarrow\; x \models^+ \varphi$ and $x \models^- \psi$ **(N4)**
  - $x \models^- \varphi \to \psi \;\Leftrightarrow\; \forall y \sqsupseteq x(x \models^+ \varphi$ implies $x \models^- \psi)$ **(C)**

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- Logic is selfextensional if it is closed under the replacement rule

$$\frac{\varphi \dashv\vdash \psi}{\chi(\varphi) \dashv\vdash \chi(\psi)} \quad \text{or} \quad \frac{\varphi \leftrightarrow \psi}{\chi(\varphi) \leftrightarrow \chi(\psi)}$$

- **N4** is not selfextensional.

$$\neg(\varphi \to \psi) \leftrightarrow \neg(\neg\varphi \vee \psi) \in \mathbf{N4}$$

$$(\varphi \to \psi) \leftrightarrow (\neg\varphi \vee \psi) \notin \mathbf{N4}$$

The Last but One Slide

Hyperintesionality = Non-selfextensionality?

The Last Slide

THANK YOU!