TWO THEOREMS ON SOLVABLE AND NILPOTENT GROUPS

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The classical Neumann-Neumann's embedding theorem [1] states that every countable solvable group G of solvable length l can be embedded in a 2-generated solvable group H of length $l+2$. On the other hand, not every countable abelian group can be embedded in a 2-generated metabelian group. Indeed, the group of rationals $\mathbb{Q}$ cannot be embedded even in a finitely generated metabelian group because by Hall's theorem every metabelian group is residually finite, but $\mathbb{Q}$ is not. Thus, the parameter $l+2$ in the general case cannot be improved. The question of the possibility of embedding a finitely generated solvable group of length l in a 2- or at least k-generated (for a fixed small k) solvable group of length $l+1$ remained open. This question was explicitly posed by V.H. Mikaelian and A.Y. Olshanskii in [2] and by A.Y. Olshanskii in [3], Question 18.73. The following theorem answers this question.

Theorem 1. *Let G be a countable group such that the abelianization $G_{ab} = G/G'$ is direct product of a free abelian group and a finite group. Then G can be embedded in a 4-generated subgroup H of the Cartesian wreath product $GWr\mathbb{Z}^3$. Thus, any finitely generated solvable group G of length l can be embedded in a 4-generated solvable group H of length $l+1$. If G is finite (periodic), then H can be found also finite (periodic).*

Remind, that any finitely generated nilpotent group is embeddable in a 2-generated nilpotent group [5], and any polycyclic group is embeddable in a 2-generated polycyclic group [6].

The following theorem answers the well-known question explicitly posed by M. Lohrey and B. Steinberg (see [4]).

Theorem 2. *The submonoid membership problem is unsolvable for a free nilpotent group N_{rl} of class $l \geq 2$ and sufficiently large rank r.*

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References

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