HOW TO PROVE, AND NOT TO PROVE, CONSISTENCY

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The consistency of a formal theory is a sequential property $C = C_0, C_1, \ldots, C_n, \ldots$, where each C_n states that the nth derivation does not contain a contradiction. For proving C in a theory T, Hilbert suggested (i) finding a procedure that given n builds a T-proof of C_n and (ii) proving in T that this procedure always works.

However, for Peano Arithmetic PA, the traditional way here has been to compress C into a single arithmetical formula Consis(PA) and apply the Second G?del Incompleteness theorem, stating the unprovability of Consis(PA) in PA, to claim the unprovability of C in PA. This chain of reasoning is fundamentally flawed: one can only conclude that (a compressed form of) consistency is not provable in a FINITE fragment of PA whereas PA is known to be (much) stronger than any of its finite fragments.

Following the original Hilbert's approach, we were able to show that the consistency property of PA is indeed provable in PA. These findings dismantle a foundational "impossibility paradigm": there exists no consistency proof of a system that can be formalized in the system itself. (Encyclopaedia Britannica, Article "Metalogic," 2000).

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