THE COMPUTATIONAL ASPECTS OF THE THEORY OF WEAK PROBABILITY SPACES

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In probability logic, a fundamental role is played by weak probability spaces, in which measures are required to be finitely additive, but not necessarily countably additive. The present talk will be concerned with the (elementary) theory of weak probability spaces and its computational aspects.

To give more details, let us fix a countable set of *variables*, intended to range over events in a given space. By a *basic μ-formula* we mean an expression of the form

$$f\,(\mu\,(\phi_1), \ldots, \mu\,(\phi_m)) \;\leqslant\; g\,(\mu\,(\phi_{m+1}), \ldots, \mu\,(\phi_{m+n}))$$

where f and g are polynomials with integer coefficients, $\phi_1, \ldots, \phi_{m+n}$ are Boolean combinations of variables. The *μ-formulas* are built up from the basic μ-formulas in the obvious way, as in first-order logic. Naturally, for a class $\mathcal{K}$ of spaces, the *theory of* $\mathcal{K}$ is the set of all μ-sentences true in every space in $\mathcal{K}$.

Next, call a μ-formula *flat* if each of its basic subformulas has the form

$$\mu\,(\phi) \;=\; \mu\,(\psi).$$

It turns out that even the flat fragment of the theory of weak probability spaces is Π^1_1-complete (as well as the full theory). Similar results can be obtained for various 'first-order' logics of probability (i.e., for languages analogous to those in [1]); see [4] and [5]. So in terms of closure ordinals (cf. [2]), many infinitary probabilistic calculi are as hard as possible: the corresponding closure ordinals coincide with ω_1^{CK}, which denotes the least non-constructive ordinal. In particular, this applies to various proof systems developed by Z. Ognjanović and his colleagues, like those in [3].

References

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