

THE DYMENT LATTICE AND INTUITIONISTIC LOGIC

A. SORBI

The *logic* $L(\mathcal{B})$ of a Brouwer algebra $\mathcal{B} = \langle B, \wedge_B, \vee_B, \rightarrow_B, \neg_B, 0_B, 1_B \rangle$ is the set of propositional formulas α that evaluate to 0_B when the propositional variables occurring in α are assigned arbitrary elements of B and the propositional connectives are interpreted as the corresponding operations of the algebra, with $\wedge_B, \vee_B, \rightarrow_B$, and $\neg_B$ interpreting, respectively, the connectives $\vee, \wedge, \rightarrow$ and $\neg$. It is well known that every $L(\mathcal{B})$ contains the intuitionistic propositional calculus Int . If $a >_B 0_B$ is an element of $\mathcal{B}$, then $B_{\leq_B a} = \{x \in B : x \leq_B a\}$ is the universe of Brouwer algebra as well, where the operations are the restrictions of the operations of $\mathcal{B}$ to $\{x : x \leq_B a\}$, except that $1_{B,a} = a$ and $\neg_{B,a}(x) = x \rightarrow_B a$.

We investigate two Brouwer algebras arising in computability theory, the Dymont lattice and the Dymont-Muchnik lattice, which can be viewed as natural generalizations, in the setting of enumeration reducibility, of the well known Medvedev and Muchnik lattice arising from Turing reducibility.

If $\mathcal{A}, \mathcal{B}$ are *Dymont mass problems*, i.e., sets of partial numerical functions, we say that $\mathcal{A}$ is *Dymont-reducible* to $\mathcal{B}$, and write $\mathcal{A} \leq_D \mathcal{B}$, if there exists an enumeration operator Φ , such that for every $\varphi \in \mathcal{B}$, $\Phi(\text{graph}(\varphi))$ outputs the graph of a partial function that lies in $\mathcal{A}$. The degrees (called *Dymont degrees*) corresponding to $\leq_D$ form not only a poset, but in fact a Brouwer algebra $\mathcal{M}_D$, called the *Dymont lattice*.

The non-uniform counterpart of the Dymont reducibility is the *Dymont-Muchnik reducibility* $\leq_{D,w}$. Thus, $\mathcal{A} \leq_{D,w} \mathcal{B}$ if and only if $(\forall \varphi \in \mathcal{B})(\exists \chi \in \mathcal{A})[\text{graph}(\chi) \leq_e \text{graph}(\varphi)]$. This reducibility gives rise to a Brouwer algebra, denoted by $\mathcal{M}_{D,w}$ and called the *Dymont-Muchnik lattice*.

Theorem 1 *There exist Dymont degrees $\mathbf{A}$ such that $L(\mathcal{M}_{D \leq_D \mathbf{A}}) = \text{Int}$.*

The proof combines algebraic and computability-theoretic methods. The algebraic part is essentially the same as that used for the Medvedev lattice, while the computability-theoretic part requires additional care in view of the substantial differences between enumeration reducibility and Turing reducibility.

Theorem 2 *There exist Dymont-Muchnik degrees $\mathbf{A}$ such that $L(\mathcal{M}_{D,w \leq_{D,w} \mathbf{A}}) = \text{Int}$.*

Here, the existence of $\mathbf{A}$ can be derived from the existence of special classes $\mathcal{A}$ of sets of numbers, called *e-splitting* classes. Proving that these classes exist is not an easy task, since the known examples of splitting classes for the Turing degrees do not readily transfer to the setting of enumeration reducibility.

Corollary *Both $L(\mathcal{M}_D)$ and $L(\mathcal{M}_{D,w})$ coincide with the intermediate logic obtained by extending Int with the axiom scheme $\neg\alpha \vee \neg\neg\alpha$.*

The research described in this abstract is joint work with Hristo Ganchev, Paul Shafer, Theodore Slaman, and Mariya Soskova.

DEPARTMENT OF INFORMATION ENGINEERING AND MATHEMATICS, UNIVERSITY OF SIENA, SIENA, ITALY

Email address: sorbi@unisi.it